Dihedral Groups & Abelian Groups

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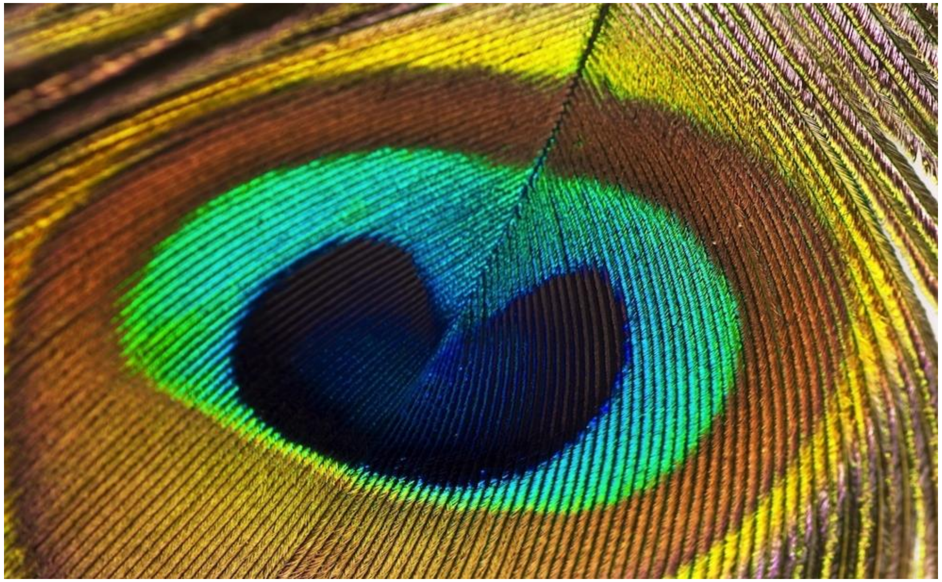
Symmetry in everyday language refers to a sense of harmonious and beautiful proportion and balance.

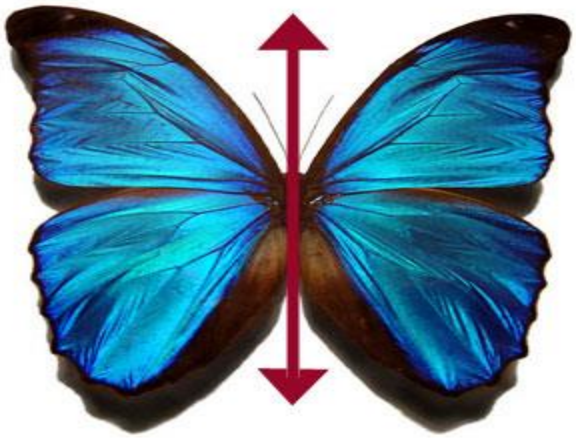


Types Of Symmetry

- ❖ Line Symmetry
- ❖ Rotational Symmetry

Line Symmetry

An object has line symmetry or reflective symmetry when it is the same on both sides of a line drawn down the middle.

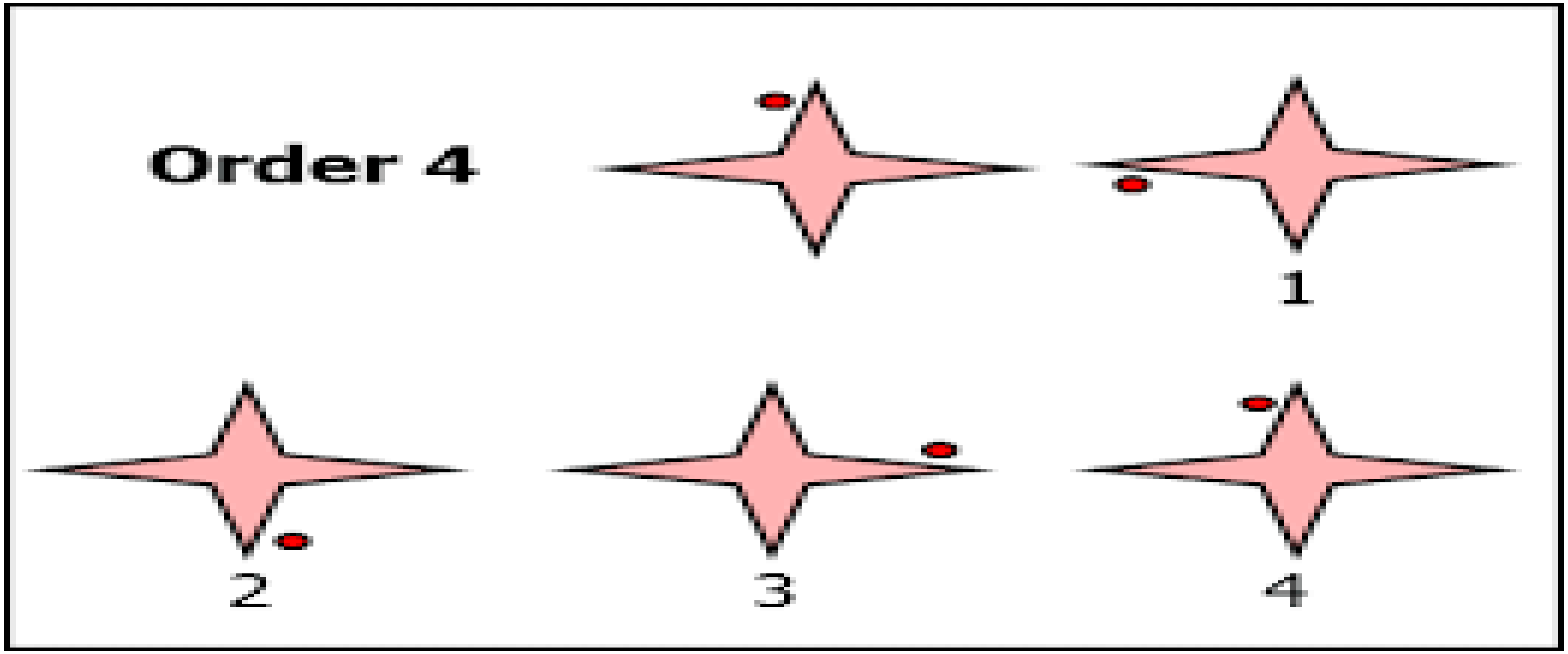


Rotational Symmetry

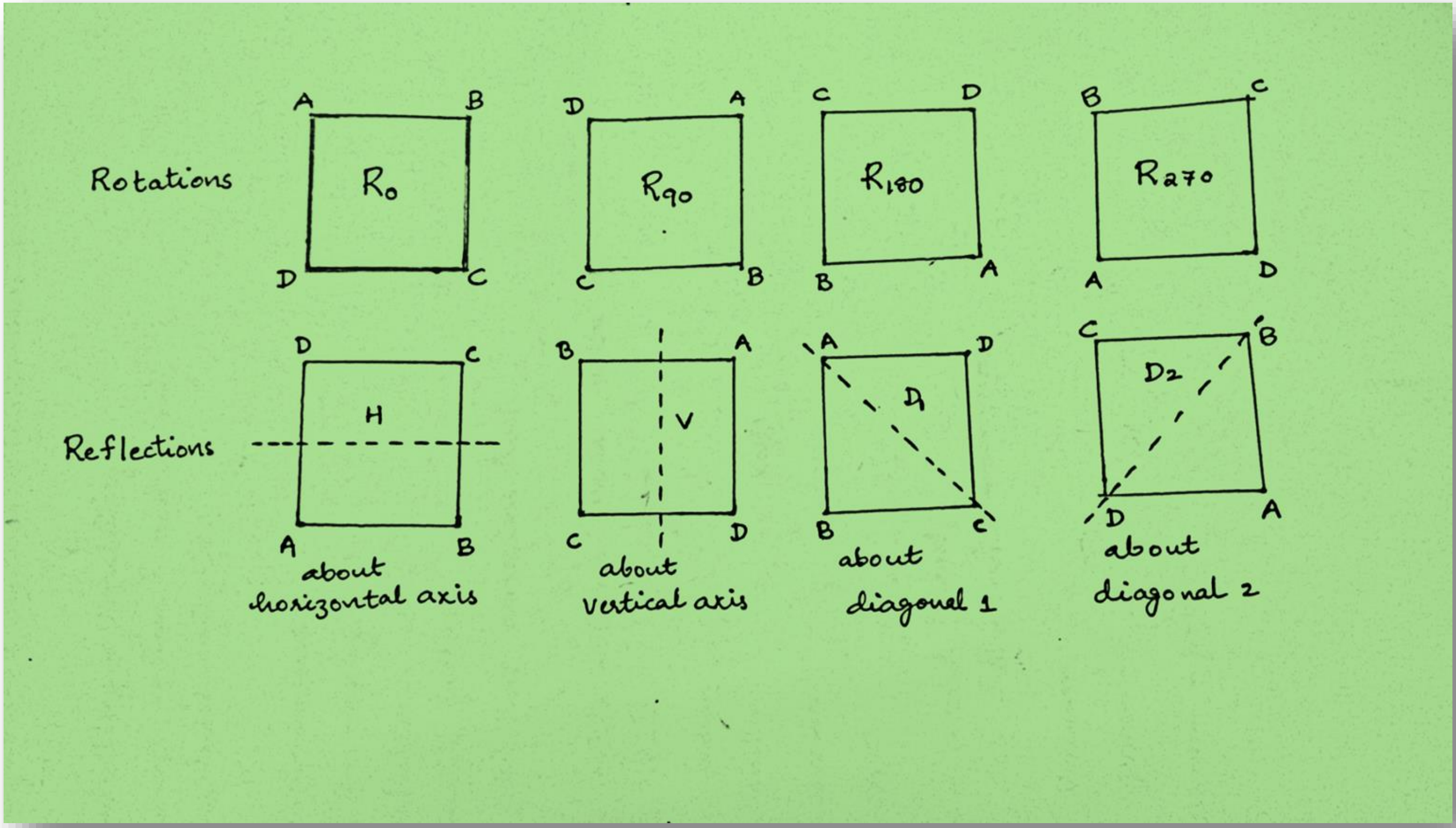
An object is said to have rotational symmetry if it remains the same after being rotated around a central point.

Order Of Rotational Symmetry

The number of positions a figure can be rotated to, without bringing in any changes to the way it looks originally, is called its order of rotational symmetry.



SYMMETRIES OF A SQUARE



Composition Of Motions

R_{90} * H = D_2

V * D_2 = R_{90}

*	R_0	R_{90}	R_{180}	R_{270}	H	V	D_1	D_2
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D_1	D_2
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D_2	D_1	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D_2	D_1
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D_1	D_2	V	H
H	H	D_1	V	D_2	R_0	R_{180}	R_{90}	R_{270}
V	V	D_2	H	D_1	R_{180}	R_0	R_{270}	R_{90}
D_1	D_1	V	D_2	H	R_{270}	R_{90}	R_0	R_{180}
D_2	D_2	H	D_1	V	R_{90}	R_{270}	R_{180}	R_0

Observations

$$\boldsymbol{D_4} = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D_1, D_2\}$$

- $a, b \in \boldsymbol{D_4}$ $then\ (a * b) \in \boldsymbol{D_4}$ (***closure***)
- $a \in \boldsymbol{D_4}$ $then\ a * R_0 = R_0 * a = a$ (***identity***)
- $\forall\ a \in \boldsymbol{D_4}$ $there\ exists\ b \in \boldsymbol{D_4}$ such that
  $a * b = b * a = R_0$ (***inverse***)
- Composition is ***associative***

Dihedral Group

- The analysis carried out for a square can similarly be done for an equilateral triangle or regular pentagon or any regular n-gon ($n \geq 3$). The corresponding group is denoted by D_n and is called the dihedral group of order 2n.

- The dihedral group consists of n rotations $\{ R_0, R_{\frac{360}{n}}, R_{\frac{2\times 360}{n}} \ldots . R_{\frac{(n-1)\times 360}{n}} \}$ and n reflections.
- The set of rotations is a cyclic subgroup of $\mathbf{D_n}$ of order n and is generated by $R_{\frac{360}{n}}$.

Applications

- Decorative designs used on floor coverings, pottery and buildings.
- Symmetry considerations are applied in orbital calculations, determining energy levels of atoms and molecules.
- Mineralogists determine internal structures of crystals from the symmetry present in the X-ray projections.
- Mathematically impossible for a crystal to possess a $\boldsymbol{D_n}$ symmetry pattern with $n \geq 5$.

ABELIAN GROUPS

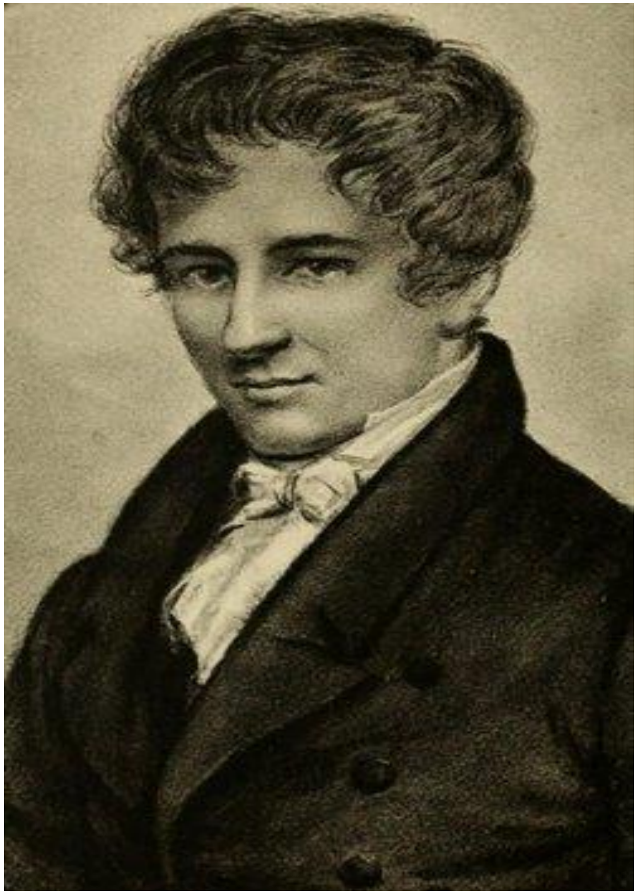
Preliminaries

- A group is a set G together with a binary operation $*$ that satisfies the following properties

  1) $*$ is associative
  2) $\exists\, e \in G$, such that $\forall g \in G,\ e * g = g * e = g$
  3) $\forall g \in G,\ \exists\, g' \in G$ such that $g' * g = g * g' = e$

- An Abelian group is a group whose binary operation is commutative.

Abelian group is named after Niels Henrik Abel, a Norwegian mathematician who made pioneering contributions in the field of Group theory.



Examples for Abelian Group

- $(\mathbb{Z}, +)$ set of integers with respect to addition.
- Every cyclic group is Abelian.
- $(\mathbb{Z}_n, +_n)$ is a cyclic group with n elements.
- Every finite cyclic group with n elements is isomorphic to $(\mathbb{Z}_n, +_n)$.

Cartesian product of Abelian groups

Let $G_1, G_2, \dots, G_n$ be Abelian groups. Then the Cartesian product

$$G_1 \times G_2 \times \cdots \times G_n = \{(a_1, a_2, \dots, a_n) : a_i \in G_i\}$$

1) $(G_1 \times G_2 \times \cdots \times G_n, *)$ is a group with respect to the operation $*$ defined by

$$(a_1, a_2, \dots, a_n) * (b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n)$$

2) $(G_1 \times G_2 \times \cdots \times G_n, *)$ is also Abelian.

THEOREM 1:

The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

DECOMPOSABLE GROUP

A group G is decomposable if it is isomorphic to a direct product of two proper nontrivial subgroups. Otherwise G is indecomposable.

THEOREM 2:

The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

THEOREM 3:

If m divides the order of a finite abelian group G then G has a subgroup of order m.

THEOREM 4:

If m is a square free integer then every abelian group of order m is cyclic.

OBSERVATIONS

- If G is an abelian group and H be the subset of G consisting of the identity e together with all elements of G of order 2.Then H is a subgroup of G.
- A finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $Z_p \times Z_p$ for some prime p.
- If a finite abelian group has order a power of a prime p then the order of every element in the group is a power of p.

NORMAL SUBGROUP

Let H be a subgroup of G then H is called a normal subgroup of G if aH=Ha, $\forall a \in G$.

OBSERVATIONS

- Every subgroup of an abelian group is normal.
- For a prime number p, every group G of order p^2 is abelian.


Thank you