C 80272 **(Pages : 3)** **Name..**

Reg. No......................................

SIXTH SEMESTER B.A./B.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCBCSS—UG)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all the twelve** *questions.*
Each question carries 1 *mark.*

1. State Fundamental theorem of Algebra.
2. State division algorithm for integers.
3. Find a value of c for which the Diophantine equation $24x + 36y = c$ has a solution.
4. Write the number of solutions of the linear congruence $18x = 30(\text{mod}\, 42)$, which are incongruent modulo 42.
5. State Fermat's Little Theorem.
6. Find $\sigma(12)$.
7. Find $\phi(30)$.
8. What do you mean by a subspace of a vector space ?
9. Define basis of a vector space.
10. Define a linear map.
11. If the mapping f : V —> W is linear then show that $f(0_v) = 0_w$.
12. Define the nullity of a linear map.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *questions from among the questions 13 to 26.*
Each question carries 4 *marks.*

13. Show that any integer of the form $6k+5$ is also of the form $3j+2$ but not conversely.
14. Prove that $\gcd(ka,kb) = k\gcd(a,b)$ where k is any positive integer.

Turn over

15. Find one integer solutions of the Diophantine equation $33x + 14y = 115$.

16. If p is a prime number and $p|ab$, then show that $p|a$ or $p|b$.

17. Prove that $\sqrt{5}$ is irrational.

18. Without performing the division, determine whether the integer 176521221 is divisible by 9 or 11.

19. Using Fermat's theorem, show that 17 divides $11^{104} + 1$.

20. Show that the function τ is multiplicative.

21. Given integers a, b, c, then prove that $\gcd(a,bc) = 1$ if and only if $\gcd(a,b) = 1$ and $\gcd(a,c) = 1$.

22. Show that the intersection of any set of subspaces of a vector space V is a subspace of V.

23. If V is a vector space over C of dimension n, prove that V can be regarded as a vector space over R of dimension $2n$.

24. Let V be a vector space of dimension $n \geq 1$ over a field F. Then show that V is isomorphic to the vector space F^n.

25. If $f : R^2 \to R^2$ is given by $f(a,b) = (b,0)$, prove that $Im f = Ker f$.

26. Let $A = \{(x,y,0); x,y \in R\}$ and let $B = \{(x,0,z); x,y,z \in R\}$, subspaces of R^2. Show that $A \simeq B$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *questions from among the questions 27 to 35.*
Each question carries 7 marks.

27. Given integers a and b with $b > 0$, prove that there exist unique integers q and r satisfying $a = bq + r, 0 \leq r < b$.

28. Suppose a and b are integers, not both zero. For a positive integer d, prove that, $d = \gcd(a,b)$ if and only if (a) $d|a$ and $d|b$; (b) Whenever $c|a$ and $c|b$ then $c|d$.

29. If $p \geq q \geq 5$ and p and q are both primes, prove that $24 | p^2 - q^2$.

30. State and prove Euclid theorem on primes.

31. Derive Legendre formula for $n!$

32. Show that a non-empty subset S of a vector space V is a basis of V if and only if every element of V can be expressed in a unique way as a linear combination of elements of S.

33. Show that (1, 1, 0, 0), (–1, –1, 1, 2), (1, –1, 1, 3), (0, 1, –1, –3) is a basis of R^4 and express a general vector (a, b, c, d) as a linear combination of these basis elements.

34. Let $f : V \longrightarrow W$ be linear. If X is a subspace of V then show that $f^{\rightarrow}(X)$ is a subspace of W; and if Y is a subspace of W then show that $f^{\leftarrow}(Y)$ is a subspace of V.

35. Let $f : R^3 \rightarrow R^2$ be linear such that $f(1, 1, 0) = (1, 2)$, $f(1, 0, 1) = (0, 0)$, $f(0, 1, 1) = (2, 1)$. Determine f completely.

(6 × 7 = 42 marks)

Section D

Answer any **two** *questions from among the questions 36 to 38.*
Each question carries 13 *marks.*

36. Prove that every positive integer $n > 1$ can be expressed as product of primes ; this representation is unique apart from the order in which factors occur.

37. State and prove Chinese Remainder Theorem.

38. Let V and W be vector spaces of finite dimension over a field F. If $f : V \longrightarrow W$ is linear then show that $\dim V = \dim \operatorname{Im} f + \dim \operatorname{Ker} f$.

(2 × 13 = 26 marks)