D 70972 (Pages : 2) Name....................................

Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION NOVEMBER 2019

(CUCSS)

Mathematics

MT 3C 14—FUNCTIONAL ANALYSIS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Define the metric space l^p, for $1 \le p \le \infty$ and the metric d_p on it. Is l^p separable for $1 \le p \le \infty$.
2. Give an element of $L^1(\mathbb{R})$ which does not belong to $L^2(\mathbb{R})$. Justify your answer.
3. Define the quotient norm on the quotient space X/Y, whereY is a closed subspace of a normed space X.
4. Let X be a convex subset of a normed space X. Show that the interior E^0 of E is convex.
5. Define a strictly convex normed space and give one example.
6. Show that BL(X, Y) is a normed space if X and Y are normed spaces.
7. State Hahn-Banach separation theorem.
8. Define the second dual of a normed space X and describe the canonical embedding of X into its double dual.
9. State the Taylor-Foguel Theorem.
10. Define a projectión on a linear space. Show thât for a projection P, R(P) = Z (I – P).
11. State open mapping theorem for Banach spaces.
12. Define a Hilbert space. Give an example of a Banach space which is not a Hilbert space.
13. State the Gram-Schmidt orthonormalization theorem.
14. State the Parseval formula in a Hilbert space.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Show that l^∞ is complete.
16. Show that the metric space $L^\infty([a,b]), a < b,$ is not separable.

Turn over

17. Show that the three norms $\| \|_1, \| \|_2$ and $\| \|_\infty$ on $\mathbb{C}^n$ are equivalent.

18. Verify whether $\mathbb{C}^n$ with norms $\| \|_1, \| \|_2$ and $\| \|_\infty$ is strictly convex or not for $n \geq 2$.

19. Prove that a Banach space cannot have a denumerable Hamel basis.

20. State and prove Resonance Theorem for a normed space.

21. Let X be a normed space in which every absolutely summable series of elements of X is summable. Prove that X is a Banach space.

22. Show that the inverse of a bijective continuous map may not be continuous.

23. State and prove the Schwarz inequality in an upper product space.

24. Let $\mathbb{H}$ be a Hilbert space, let $\{u_n : n \in \mathbb{N}\}$ be a countable orthonormal set in $\mathbb{H}$ and let $k_n \in \mathbb{K}$ such that $\sum_n |k_n|^2 < \infty$. Show that $\sum_n k_n u_n$ converges in $\mathbb{H}$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. State and prove Uniform Boundedness Principle for Banach spaces.

26. Let T be a set and let X be a subspace of B(T) with sup norm, $1 \in$ X and f be a linear functional on X. Prove the following :

    (a) If f is continuous and $\|f\| = 1$, then f is positive.

    (b) If Re $x \in$ X whenever $x \in$ X and if f is positive, then f is continuous and $\|f\| = f(1)$.

27. Prove that the following three conditions are equivalent for a non-zero Hilbert space $\mathbb{H}$ over $\mathbb{K}$:

    (a) $\mathbb{H}$ has a countable orthonormal basis.

    (b) $\mathbb{H}$ is linear isometric to $\mathbb{K}^n$ for some $n \in \mathbb{N}$ or to l^2.

    (c) $\mathbb{H}$ is separable.

28. Let $\{u_\alpha\}$ be an orthonormal set in a linear product space X and $x \in$ X. Let $E_x := \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Prove the following :

    (a) E_x is countable.

    (b) If E_x is not finite, then $\langle x, u_n \rangle \to 0$ as $n \to \infty$

(2 × 4 = 8 weightage)