D 72971 (Pages : 4) Name..................................

Reg. No..............................

FIRST SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION DECEMBER 2019

(CBCSS)

Mathematics

MTH 1C 03—REAL ANALYSIS—I

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Prove that every neighborhood is an open set.

2. Let E be a subset of a metric space. Prove that E is closed if and only if $\bar{E} = E$.

3. Prove that continuous image of a compact metric space is compact.

4. Let f be a differentiable function in (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then prove that f is a constant.

5. Show by an example that L' Hospital's rule need not hold for complex valued functions.

6. Let f be a bounded real function such that f^2 is Riemann integrable on $[a, b]$. Is f Riemann integrable on $[a, b]$? Justify your answer.

7. Let γ be a curve in the complex plane, defined on $[0, 2\pi]$ by :

$$\gamma(t) = e^{2it}.$$

Prove that the length of γ is 4π.

8. Let $\{f_n\}$ be a sequence of Riemann integrable functions such that $f_n \to f$. Is f Riemann integrable ? Justify your answer.

(8 × 1 = 8 weightage)

Turn over

Part B

Unit I

Answer any **two** *questions from each unit.*
Each question carries a weightage 2.

9. Prove that every infinite subset of a countable set A is countable.

10. Prove that a mapping f of a metric space X into metric space Y is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

11. Let f be a monotonic function on (a, b). Prove that the set of points of (a, b) at which f is discontinuities is atmost countable.

(2 × 2 = 4 weightage)

Unit II

12. Let f be a real differentiable function on $[a, b]$ and let $f'(a) < \lambda < f'(b)$. Prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

13. Let f be a bounded function and α be a monotonic increasing function on $[a, b]$. If P_1 is a refinement of P, then prove that :

$$U(P_1, f, \alpha) \leq U(P, f, \alpha).$$

14. For $1 < s < \infty$, define :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that $\zeta(s) = s \int_1^s \frac{[x]}{x^{s+1}}\, dx.$

(2 × 2 = 4 weightage)

Unit III

15. On what intervals does the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$ converge uniformly ?

16. Let C (X) denote the set of all complex valued, continuous, bounded functions defined on a metric space X. Prove that C (X) is a complete metric space with respect to the metric

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

17. If K is compact, $f_n \in C(K)$ for $n = 1, 2, 3, \ldots$ and if $\{f_n\}$ is point-wise bounded and equicontinuous on K, then prove that $\{f_n\}$ is uniformly bounded on K.

(2 × 2 = 4 weightage)

Part C

Answer any **two** *from the following four questions* (18-21).
Each question has weightage 5.

18. (a) Define perfect set and give an example of it.

(b) Let E be a subset of $\mathbb{R}^k$. Prove that the following are equivalent :

(i) E is closed and bounded.

(ii) E is compact.

(iii) Every infinite subset of E has a limit point in E.

19. (a) Let f be continuous on $[a, b]$ and $f'(x)$ exists at some point $x \in (a, b)$. If g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$, then prove that the function h defined on $[a, b]$ by :

$$h(t) = g(f(t))$$

is differentiable at x and

$$h'(x) = g'(f(x)) f'(x).$$

(b) If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then prove that there is a point $x \in (a, b)$ such that :

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

20. (a) Let f be a bounded, monotonic increasing real function and α be a continuous, monotonic increasing real function on $[a, b]$. Prove that f is Riemann-Stieltjes integrable with respect to α on $[a, b]$.

(b) Let f be a bounded real function on $[a, b]$. If f is Riemann integrable on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then prove that

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

21. If f is a continuous complex function on $[a, b]$, then prove that there exists a sequence of polynomials P_n such that

$$\lim_{x \to \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$.

(2 × 5 = 10 weightage)