D 72783 (Pages : 3) Name..............................

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FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2019

(CUCSS)

Mathematics

MT1C02—LINEAR ALGEBRA

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *the questions.*
Each question carries weightage of 1.

1. Let V be a vector space over field K. Show that for any scalar $k \in K$ and $0 \in V, k0 = 0$.

2. Consider $V = \mathbb{R}^3$ as a vector space over $\mathbb{R}$. Show that W is not a subpsace of V, where $W = \{(a, b, c) : a \geq 0\}$.

3. For which value of k will the vector $u = (1, -2, k)$ in $\mathbb{R}^3$ be a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$?

4. Show that the mapping $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = xy$ is not linear.

5. Let ϕ be the linear functional on $\mathbb{R}^2$ defined by $\phi(2,1) = 15$ and $\phi(1, -2) = -10$. Find $\phi(x, y)$.

6. Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Find all eigenvalues and the corresponding eigenvectors of A viewed as a matrix over the real field $\mathbb{R}$.

7. Show that 0 is an eigenvalue of T if and only if T is singular.

8. Show that the vectors $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0 \ldots\ldots, 0), e_3 = (0, 0, 1, \ldots 0), \ldots, e_3 = (0\ 0, 0, \ldots, 1)$ is a basis for $\mathbb{R}^n$.

9. State the cayley-Hamilton Theorem.

Turn over

10. For the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$, find a polynomial having the matrix A as a root.

11. Consider the three polynomials defined as follows :

$p_1(t) = 7t^5 - 4t^2 + 3,\ p_2(t) = 2t^5 + 5t^2,\ p_3(t) = 8t^5 - 23t^2 + 6,$. Is the set $\{p_1, p_2, p_3\}$ linearly independent ?

12. Let $\mathbb{R}^3$ be equipped with the standard inner product. What is the orthogonal proejction of the vector $x = (1, 2, 3)$ onto the vector $y = (3, 2, 1)$?

13. Let V be the set of polynomials with the inner product $\langle f, g \rangle = \int_0^1 f(t)\, g(t)$. Let $f(t) = t + 2$. Find $\| f \|$.

14. If u is orthogonal to v, then show that every scalar multiple of u is also orthogonal to v.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries weightage of 2.

15. Let V be the vector space of n-square matrices over a field $\mathbb{R}$. Let U and W be the subspaces of symmetric and antisymmetric matrices respectively. Show that $V = U \oplus W$.

16. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by

$T(x, y, s, t) = (x - y + s + t, x + 2s - t, x + y + 3s - 3t)$. Find a basis and the dimension of the image U of T.

17. Let V be a vector space over the field F. Prove that the intersection of any collection of subspaces of V is a subspace of V.

18. Let W be an invariant subspace for T. Show that the characteristic polynomial for the restriction operator T_W on W divides the characteristic polynomial for T.

19. State and prove the Bessel's inequality.

20. Find a unit vector orthogonal to $v_1 = (1, 1, 2)$ and $v_2\ (0, 1, 3)$ *in* $\mathbb{R}^3$.

21. For what value of k, $\langle u, v\rangle = x_1y_1 - 3x_1y_2 - 3x_2y_1 + kx_2y_2$, where $u = (x_1, x_2)$, $v = (y_1, y_2)$ is an inner product in $\mathbb{R}^2$?

22. State and Prove Cauchy-Schwarz inequality.

23. Verify that the following is an inner product in

    $\mathbb{R}^2 : \langle u, v\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$, where $u = (x_1, x_2)$, $v = (y_1, y_2)$.

24. If $\lambda \in \mathbf{F}$ is a characteristic value of a linear operator T on a vector space V, then show that for any polynomial $f(x)$ over $\mathbf{F}$, $f(\lambda)$ is a characteristic value of $f(\mathrm{T})$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage of 4.

25. If W_1 and W_2 are finite dimensional subspaces of a vector space V, then prove that $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2)$.

26. Prove that a linear operator $T : V \to V$ has a diagonal matrix representation if and only if its minimal polynomial $m(t)$ is a product of distinct linear polynomials.

27. Let V and W be vector space over the field F, and let T be a linear transformation from V into W, The null space of T^t (transpose of T) is the annihilator of the range of T. If V and W are finite dimensional, then prove that (i) $rank\left(T^t\right) = rank\,(T)$, (ii) the range of $\left(T^t\right)$ is the annihilator of the null space of T.

28. Apply Gram-Schmidt process to the vectors $w_1 = (1, 0, 1, 0)$ $w_2 = (1, 1, 1, 1)$ and $w_3 = (0, 1, 2, 1)$ to compute an orthonormal set in $\mathbb{R}^4$.

(2 × 4 = 8 weightage)