D 110208 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FIFTH SEMESTER (CBCSS—UG) DEGREE EXAMINATION NOVEMBER 2024

Mathematics

MTS 5B 05—ABSTRACT ALGEBRA

(2020 Admission onwards)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Ceiling is 25.

1. Let n be a positive integer. Prove that the congruence class $[a]_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if $(a, n) = 1$.

2. Make multiplication table for $\mathbb{Z}_6$.

3. Find the order of the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{pmatrix}$.

4. Let G be a nonempty set with an associative binary operation in which the equations $ax = b$ and $xa = b$ have solutions for all $a, b \in G$. Prove that G is a group.

5. Let G be group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

6. Prove that any group of prime order is cyclic.

7. In $GL_2(\mathbb{R})$, find the order of $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

8. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.

9. Let G be a cyclic group. If G is infinite, prove that $G \cong \mathbb{Z}$.

Turn over

10. Prove that the set of all even permutations of S_n is a subgroup of S_n.

11. Let $\phi : G_1 \to G_2$ be group homomorphism, with $K = \ker(\phi)$. Prove that K is a subgroup of G_1 such that $gKg^{-1} \in K$ for all $k \in K$ and $g \in G_1$.

12. Let $G = \mathbb{Z}_{12}$ and $H = \langle| 4 |\rangle$. Find all cosets of H.

13. State First isomorphism theorem.

14. Let G be a group. Prove that $\text{Aut}(G)$ is a group under composition of functions.

15. Prove that any subring of a field is an integral domain.

Section B

Answer any number of questions.
Each question carries 5 *marks.*
Ceiling is 35.

16. State and prove Euler theorem.

17. On $\mathbb{R}^2$, define $(a_1, a_2) \sim (b_1, b_2)$ if $a_1^2 + a_2^2 = b_1^2 + b_2^2$. Check that this defines an equivalence relation. What are the equivalence classes ?

18. Prove that the units of $\mathbb{Z}_8$ forms a group under multiplication of congruences.

19. Let G be a group with identity element e, and let H be a subset of G. Prove that H is a subgroup of G if and only if the following conditions hold :

    (a) $ab \in H$ for all $a, b \in H$;

    (b) $e \in H$; and

    (iii) $a^{-1} \in H$ for all $a \in H$.

20. Let G be a group, and let H and K be subgroups of G. If $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, Prove that HK is a subgroup of G.

21. Prove that every subgroup of a cyclic group is cyclic.

22. State and prove fundamental theorem of homomorphism.

23. Let G be a group with normal subgroups H, K such that $HK = G$ and $H \cap K = \{e\}$. Prove that $G \cong H \times K$.

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*
Maximum 20 *marks.*

24. a) Prove that every permutation in S_n can be written as a product of disjoint cycles

b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 7 & 6 & 3 & 8 & 1 & 4 \end{pmatrix}$ be a permutation in S_8. Express σ as a product of disjoint cycles.

25. a) Let $\phi : G_1 \to G_2$ be an isomorphism of groups. Prove that ϕ preserves following structural properties :

(i) If a has order n in G_1, then $\phi(a)$ has order n in G_2,

(ii) If G_1 is abelian, then so is G_2,

(iii) If G_1 is cyclic, then so is G_2.

b) Prove that $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

26. Let H be a subgroup of the group G. Prove that the following conditions are equivalent.

a) H is a normal subgroup of G;

b) $aH = Ha$ for all $a \in G$;

c) for all $a, b \in G$, abH is the set theoretic product $(aH)(bH)$;

d) for all $a, b \in G$, $ab^{-1} \in H$ if and only if $a^{-1}b \in H$.

27. State and prove second isomorphism theorem.

(2 × 10 = 20 marks)