QP Code:P25C008

Reg. No : ....
Name : ....

ST MARY'S COLLEGE (AUTONOMOUS), THRISSUR-20

III SEMESTER (CBCSS -PG) DEGREE EXAMINATIONS, OCTOBER 2025
M. Sc. Mathematics
MTH3C11 : MULTIVARIABLE CALCULUS & GEOMETRY
2024 Admission Onwards

Time: 3 Hours **Maximum Weightage: 30**

Part A

*Answer **all** questions. Weightage **1** for each question.* ***(8x1 = 8 Weightage)***

1. For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ define $\|A\|$. Prove that $|A(x)| \leq \|A\|\,|x|$, $\forall x \in \mathbb{R}^n$.
2. Check whether the set of vectors $\{(1,1,2),(3,-1,1),(4,3,2),(2,1,1)\}$ is linearly independent.
3. Find the tangent vector of $\gamma(t) = (e^t, t^2)$.
4. Check whether the given curve is a unit speed curve $\gamma(t) = (\frac{4}{5}\cos t, 1 - \sin t, \frac{-3}{5}\cos t)$
5. Compute the curvature of $\gamma(t) = (\frac{4}{5}\cos t, 1 - \sin t, \frac{-3}{5}\cos t)$.
6. Find the first fundamental form of $\sigma(u,v) = (u, v, u^2 + v^2)$.
7. What is normal curvature and geodesic curvature of a unit speed curve γ on an oriented surface S?
8. Find the second fundamental form of $\sigma(u,v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$.

Part B

*Answer **any two** questions from each module. Weightage **2** for each question.*
(6x2 = 12 Weightage)

Unit-I

9. Prove that $f'(x)(ej) = \sum_{i=1}^{m} D_j f_i(x)$, where $f : E \to \mathbb{R}^m$, f is differentiable at $x \in E$
10. Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that $dim X \leq r$.
11. Suppose f maps an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$. If $f \in \mathscr{C}'(E)$ then prove that the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$

Turn Over

Unit-II

12. Compute τ, t, n, b for $\gamma(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{\sqrt{t}}{2}\right)$.

13. Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ then prove that $\kappa = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$.

14. Prove that open subset of a smooth surface is a smooth surface.

Unit-III

15. Let $\sigma(u, v)$ be a surface patch with standard unit normal $\mathbf{N}(u, v)$. Prove that $\mathbf{N}_u \cdot \sigma_u = -L,\ \mathbf{N}_u \cdot \sigma_v = \mathbf{N}_v \cdot \sigma_v = -M,\ \mathbf{N}_v \cdot \sigma_v = -N$

16. Prove that $H = \frac{LG-2MF+NE}{2(EG-F^2)},\ K = \frac{LN-M^2}{EG-F^2}$ where K is the Gaussian curvature and H is the mean curvature.

17. Find the Gaussian curvature of $\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$

Part C

*Answer **any two** questions. Weightage **5** for each question.* ***(2x5 = 10 Weightage)***

18. Prove the following
(i)There exist open sets U and V in $\mathbb{R}^n$ such that $a \in U, b \in V$, f is one-to-one on U, and $f(U) = V$.
(ii)If g is the inverse of f defined in V by $g(f(x)) = x$ then $g \in \mathscr{C}'(V)$
where f is a $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$.

19. Let $k : (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Then prove the following
(i) There is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k.
(ii)If $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, there is a direct isometry M of $\mathbb{R}^2$ such that $\tilde{\gamma} = M(\gamma(s)), \forall s \in (\alpha, \beta)$

20. (i) Define derivative $D_p f$ of a smooth map f at a point p between smooth surfaces.
(ii) If S_1, S_2 and S_3 are surfaces and $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are smooth maps, then prove that for all $p \in S_1$,$D_p(f_2 o f_1) = D_{f_1(p)} f_2 o D_p f_1$
(iii) If $f : S_1 \to S_2$ is a diffeomorphism, then prove that for all $p \in S_1$ the linear map $D_p f : T_p S_1 \to T_{f(p)} S_2$ is invertible.

21. Show that
(i) $\kappa_n = \ddot{\gamma} \cdot \mathbf{N}, \kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma}), \kappa^2 = \kappa_n^2 + \kappa_g^2$
(ii) $\kappa_n = \kappa\cos\psi, \kappa_g = \pm\kappa\sin\psi$
where κ is the curvature of γ and ψ is the angle between $\mathbf{N}$ and the principal normal $\mathbf{n}$ of γ .

****