By using this we can construct a sequence $\{g_n\}$ of simple functions such that $\int_a^b \|f - g_n\| d\alpha < \frac{1}{n}$ and $\|g_n\| \leq \|f\| + \frac{1}{n}$ for every $n \geq 1$.
Also, since $\lim_{n\to\infty} \int_a^b \|f - g_n\| d\alpha = 0$, we can choose a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to f$.
Now, g_{n_k} are integrable and $\|g_{n_k}\| \leq \|f\| + \frac{1}{n_k}$ by dominated convergence theorem, f is Henstock-Stieltjes integrable with respect to α.
On the contrary, suppose f is Henstock-Stieltjes integrable with respect to α. Then f is measurable and hence $\|f\|$ is Lebesgue-Stieltjes integrable with respect to α. Consequently, f is Bochner-Stieltjes integrable with respect to α. □

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FUZZY BOUNDED LINEAR MAPS AND QUOTIENT SPACES

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Abstract. In this paper fuzzy norm on a real or complex vector space is introduced. Fuzzy normed space and fuzzy Banach space are defined. Boundedness of a linear transformation in F-normed space (fuzzy normed space) is also difined i.e., F-boundedness. The quotient space of an F-normed space is defined. It is proved that $B(X, X')$, the set of all bounded linear transformation from F-normed space $(X, N, *)$ to $(X', N, *)$ is a F-Banach space if X' is a F-Banach space. It is also proved that if X is F-Banach then so is the quotient space X/M, where M is a closed linear subspace of X.

1. Introduction

I have defined F-normed space, F-Banach space, Cauchy sequence and convergence of a sequence in F-normed space, F-Banach space F-boundedness of a linear transformation on F-normed space and quotient space of F-normed space. We prove that $B(X, X')$ the set of all bounded linear transformations from the F-normed space $(X, N, *)$ to the F-normed space $(X, N, *)$ is F-normed space under the F-norm $N(T, t) = \text{Inf}\{N(T(x), t)/x \in X\}$ and with respect to the pointwise linear operation defined by $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(\alpha T)(x) = \alpha T(x)$. We have proved that if X' is F-Banach space then $B(X, X')$ is also a F-Banach space. We have also proved that if M is a closed linar subspace of an F-normed space $(X, N, *)$, then the quotient space X/M is F-normed space with the F-norm of each coset $x + M$ defined as $N(x + M, t) = \sup\{N(x + m, t)/m \in M\}$. If X is F-Banach then X/M is also F-Banach with the above F-norm.

2. Preliminary results

DEFINITION 2.1. [1] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called a t-norm if it satisfies the following conditions:

1. It is associative and commutative.
2. $a * 1 = a$
3. $a * b \leq c * d$ whenever $a \leq b$ and $b \leq d$, $a, b, c, d \in [0,1]$.

DEFINITION 2.2. A 3-tuple $(X, N, *)$ is said to be a F-normed space if X is a real or complex vector space, $*$ is a continuos t-norm and N is a function on $X \times (0, \infty)$ satisfying the following conditions:

1. $N(x,t) > 0$
2. $N(x,t) = 1$ iff. $x = 0$
3. $N(kx,t) = .N(x, t/|k|)$
4. $N(x,t) * N(y,s) \leq N(x+y, t+s)$
5. $N(x,.) : (0,\infty) \rightarrow [0,1]$ is continuous. $x, y, z \in X, t, s > 0$

DEFINITION 2.3. A sequence $\{x_n\}$ in a f-normed space $(X, N, *)$ is said to be convergent to an element $x \in X$ iff given $t > 0, 0 < r < 1$, there exists an $n_0 \in J$ (the set of all positive integers) such that $N(x_n - x, t) > 1 - r$, for every $n \geq n_0$.

DEFINITION 2.4. A sequence $\{x_n\}$ in a f-normed space $(X, N, *)$ is said to be F-Cauchy sequence if and only if for every ϵ such that $0 < \epsilon < 1$, $t > 0$, there exists $n_0 \in J$ such that

$$N(x_n - x_m, t) > 1 - \epsilon \text{ for every } n, m \geq n_0.$$

Equivalently a sequence $\{x_n\}$ in a f-normed space $(X, N, *)$ is said to be f-cauchy sequence iff given $0 < \epsilon < 1, t > 0$ there exists $n_0 \in J$ such that $N(x_{n+p} - x_m, t) > 1 - \epsilon$ for every $n \geq n_0, p > 0$.

DEFINITION 2.5. A F-normed space $(X, N, *)$ is said to be complete if every F-Cauchy sequence in X converges in X. A complete f-normed space is called F-banach space.

DEFINITION 2.6. A transformation T from the F-normed space $(X, N, *)$ to the F-normed space $(X', N', *)$ is said to be continuous at $x \in X$ if given $r_1, t_1 > 0$, $0 < r_1 < 1$, there exists $r_x, t_x > 0, 0 < r_x < 1$ such that

$$N(x-y, t_x) > 1 - r_x \Rightarrow N'(T(x) - T(y), t_1) > 1 - r_1,$$

for every $y \in X$. If T is continuous at each point $x \in X$ then it is said to be continuous on X.

DEFINITION 2.7. A linear transformation T from the F-normed space $(X, N, *)$ to the F-normed space $(X'N', *)$ is said to be f-bounded if there exists an r, $0 < r < \infty$ such that

$$N'(T(x), t) \geq N(x, t/r) \text{ for every } x \in X.$$

THEOREM 2.8. *Let $B(X, X')$ denote the set of all bounded linear transformation from $(X, N, *)$ to $(X', N, *)$. Then under the minimum t-norm $B(X, X')$ is a F-normed space with respect to the pointwise linear operations defined by $(T_1+T_2)(x) = T_1(x) + T_2(x)$ and $(\alpha T)(x) = \alpha T(x)$ and the norm defined by*

$$N(T, t) = \inf\{N(T(x), t)/x \in X\}.$$

If X' is a F-Banach space then $B(X, X')$ is also a F-Banach space.

Proof. Let $T_1, T_2 \in B(X, X')$ and α a scalar, then

$$\begin{aligned} N((T_1+T_2)(x), t) &= N(T_1(x) + T_2(x), t) \\ &\geq N(T_1(x), t/2) * N(T_2(x), t/2) \end{aligned}$$

Since T_1 and T_2 are F-bounded we have an M_1 and M_2 such that $N(T_1(x), t/2) > (x, t/2M_1)$ and $N(T_2(x), t/2) > N(x, t/2M_2)$. If $M_1 < M_2$ we have

$$\begin{aligned} N((T_1+T_2)(x), t) &\geq N(x, t/2M_1) * N(x, t/2M_2) \\ &> N(x, t/2M_2) * N(x, t/2M_2) \\ &= N(x, t/2M_2) \Rightarrow T_1 + T_2 \end{aligned}$$

is F-bounded, since $*$ is a minimum t-norm.

If T is F-bounded, then

$$\begin{aligned} N(\alpha T)(x), t) &= N\alpha(T(x), t) \\ &= N(T(x), t/|\alpha|) \\ &> N(x, t/M|\alpha|) \end{aligned}$$

for every $x \in X$, as T is F-bounded.

$$\Rightarrow \alpha T \text{ is } F\text{-bounded.}$$

Therefore $B(X, X')$ is a linear space.

Next we prove that $N(T,t) = \text{Inf}\{N(T(x),t)/x \in X\}$ is a F-norm. Let

$$\begin{aligned} N(T,t) &= 1 \\ &\Rightarrow \text{Inf}\,\{N(T(x),t)/x \in X\} = 1 \\ &\Rightarrow N(T(x),t) \geq 1 \text{ for every } x \in X \\ &\Rightarrow N(T(x),t) = 1 \text{ as we have } N(T(x),t) \leq 1 \\ &\Rightarrow T(x) = 0 \\ &\Rightarrow T = 0 \end{aligned}$$

Suppose $T = 0$,

$$\begin{aligned} &\Rightarrow T(x) = 0 \text{ for every } x \\ &\Rightarrow N(T(x),t) = 1 \text{ for every } x \in X \\ &\Rightarrow \inf\{N(T(x),t)/x \in X\} = 1 \\ &\Rightarrow N(T,t) = 1 \\ N(kT,t) &= \inf\{N(kT(x),t/x \in X\} \\ &= \inf\{N(T(x),t/|k|)/x \in X\} \\ &= N(T,t/|k|) \\ N(T_1+T_2,t+s) &= \inf N((T_1+T_2)(x),t+s)/x \in X \\ &= \inf\{N(T_1(x)+T_2(x),t+s/x \in X\} \\ &\geq inf\{N(T(x),t) * N(T_2(x),s)/x \in X\} \\ &= \inf\{N(T_1(x),t)/x \in X\} * \inf\{N(T_2(x),s)/x \in X\} \\ &= N(T_1,t) * N(T_2,s) \end{aligned}$$

This proves that $N(T,t) = \inf\{N(T(x),t)/x \in X\}$ is a F-norm on $B(X,X')$. We assume that X' is a F-Banach space and we prove that $B(X,X')$ is a F-Banach space.

Let $\{T_n\}$ be a Cauchy sequence in $B(X,X')$. Given $\epsilon > 0$, there exists an n_0 such that $N(T_n - T_m, t) > 1-\epsilon$ for every $m, n \geq n_0$. Let $x \in X$

$$\begin{aligned} N(T_n - T_m,t) &= \inf\{N(T_n - T_m)(x),t/x \in X\} \\ &\leq N(T_n - T_m)(x),t) \text{ for every } x \in X \\ \text{i.e., } N(T_n - T_m)(x),t) &\geq N(T_n - T_m,t) \text{ for every } x \in X. \\ &> 1-\epsilon \text{ for every } n,m \geq n_0 \\ &\Rightarrow \{T_n(x)\} \text{ is a Cauchy sequence in } X'. \\ &\Rightarrow \{T_n(x)\} \to T(x), \text{ as } X' \text{ is complete.} \end{aligned}$$

i.e.,

$$\lim_{n\to\infty}\{N(T_n(x) - T(x)),t)\} \to 1$$

$$\begin{aligned} N(T(x),t) &= N(T(x)+T_n(x)-T_n(x),t) \\ &\geq N(T_n(x),t/2) * N(T(x)-T_n(x),t/2) \\ &> N(T(x)-T_n(x),t/2) * N(x,t/2M) \end{aligned}$$

Taking limits on both sides,

$$\begin{aligned} N(T(x),t) &> 1 * N(T(x),t/2M) \\ &= N(x,t/2M) \\ &\Rightarrow T \text{is } F\text{-bounded} \\ T_n(\alpha x+\beta y) &= \alpha T_n(x) + \beta T_n(y). \end{aligned}$$

Taking limits on both sides we get,

$$\begin{aligned} T(\alpha x+\beta y) &= \alpha T(x) + \beta T(y) \\ &\Rightarrow T \text{ is linear. Thus } T \in B(X,X'). \end{aligned}$$

Finally we show that $T_n \to T$.
i. e., let t, ϵ be given, since $\{T_n(x)\} \to T(x)$,

$$\{N(T_n(x) - T(x)),t)\} > 1-\epsilon \text{ for every } n \geq n_x$$

Let $n_0 = \sup\{n_x/x \in X\}$. Therefore,

$$\begin{aligned} \{N(T_n(x) - T(x)),t)\} &> 1-\epsilon \text{ for all } x \in X, n \geq n_0 \\ &\Rightarrow \inf\{N(T_n(x)-T(x)),t)\} > 1-\epsilon \text{ for every } n \geq n_0 \end{aligned}$$

$N(T_n - T, T) > 1-\epsilon$ for every $n \geq n_0$

$$\Rightarrow T_n \to T.$$

Thus $B(X,X')$ is a F-Banach space. □

THEOREM 2.9. *Let M be a closed linear subspace of an F-normed space $(X,N,*)$. Then the quotient space X/M is a F-normed space with the F-norm defined by $N(x+M,t) = \sup\{N(x+m,t)/m \in M\}$. If X is a F-Banach space then X/M is also a F-Banach space.*

Proof. It can be easily verified that X/M is a linear space under the operations defined by $(x+M)+(y+M) = (x+y)+M, \alpha(x+M) = \alpha x + \in M, x, y \in X$ and α is a scalar. Now we prove that X/M is an F-normed space. The first three properties of the F-norm are trivial. We only verify the fourth property.

$$\begin{aligned}
N(x+y+M,t+s) &= \sup\{N(x+y+m,t+s/m\in M\} \\
&= \sup\{N(x+m_1+y+m_2,t+s)/m_1,m_2\in M\} \\
&= \sup\{N(x+m_1,t)*N(y+m_2,s)/m_1,m_2\in M\} \\
&= \sup\{N(x+m_1,t)/m_1\in M\}*\sup\{N(x+m_2,t)/m_2\in M\} \\
&= N(x+M,t)*N(y+M,s)
\end{aligned}$$

The fifth property of the F-norm is trivial.
We call X/M the fuzzy quotient space of X by M.
Next new will prove that X/M is F-Banach if X is F-Banach. Let $s_n\in X,(s_n+M)$ be a F-cauchy sequence in X/M. We shall extract a convergent subsequence from this Cauchy sequence as follows:

Given $\in=\frac{1}{2}, t=\frac{1}{2}$, we can find a positive integer n_1 such that

$$N((s_{n+p}+M)-(s_n+M),\frac{1}{2})>1-\frac{1}{2},\forall n\geq n_1.\ \text{Let } s_{n_1}=x_1 \tag{2.1}$$

Given $\in=\frac{1}{3}, t=\frac{1}{3}$ we can find a positive integer $n_2>n_1$ such that

$$N((s_{n+p}+M)-(s_n+M),\frac{1}{3})>1-\frac{1}{3},\ \forall n\geq n_2.\ \text{Let } s_{n_2}=x_2. \tag{2.2}$$

Since $n_2>n_1$, x_1 and x_2 will satisfy

$$N((x_2+M)-(x_1+M),\frac{1}{2})>1-\frac{1}{2}. \tag{2.3}$$

Having choosen $n_1,n_2,\ldots n_k$ and $x_1,x_2,\ldots,x_k$ we find $n_{k+1}>n_k$ such that

$$N((s_{n+p}+M)-(s_n+M),1/k+2)>1-1/k+2\forall n\geq n_{k+1}$$

and let $s_{n_{k+1}}=x_{k+1}$.

Hence we have obtained a subsequence $\{x_k+M\}$ of $\{s_n+M\}$ such that

$$N((x_{k+1}+M)-(x_k+M),1/k+1)>1-1/k+1\quad \forall k=1,2,3,\ldots$$

We shall prove this subsequence converges to an element in X/M.
Let $y_1\in x_1+M$, then $y_1=x_1+m_1, m_1\in M$. From (2.3) we get

$$N((x_2+M)-(x_1+M),\frac{1}{2})>1-\frac{1}{2},\Rightarrow \sup N(\{x_1-x_2+m,\frac{1}{2}/m\in M\}>1-\frac{1}{2},$$

implies there exists m_0 such that

$$N(x_1-x_2+m_0,\frac{1}{2})>1-\frac{1}{2},\Rightarrow N(x_1+m_1)-(x_2+m_1-m_0),\frac{1}{2})>1-\frac{1}{2}.$$

Choose y_2 as $x_2+m_1-m_0$.

$$\text{Therefore, } N(y_1-y_2,\frac{1}{2})>1-\frac{1}{2}.$$

In a similar way choose $y_3\in x_3+M$ such that $N(y_2-y_3,\frac{1}{3})>1-\frac{1}{3}$.

Continuing this manner we obtain a sequence $\{y_n\}$ in X such that

$$N(y_n-y_{n+1},\frac{1}{n+1})>1-\frac{1}{n+1}.$$

We shall show that $\{y_n\}$ is a F-Cauchy sequence in X.

$$\begin{aligned}
N(y_{n+q}-y_n,t) &= N(y_n-y_{n+1}+y_{n+1}-y_{n+2}+y_{n+2}-\cdots+y_{n+q-1}-y_{n+q},t) \\
&\geq N(y_n-y_{n+1},\frac{t}{q})*N(y_{n+1}-y_{n+2},\frac{t}{q})*N(y_{n+2}-y_{n+3},\frac{t}{q})*\cdots \\
&\quad \cdots*N(y_{n+q+1}-y_{n+q}\frac{t}{q})
\end{aligned}$$

choose $\frac{t}{q}>\frac{1}{n+1}$ then

$$\begin{aligned}
N(y_{n+q}-y_n,t) &\geq N(y_n-y_{n+1},\frac{1}{n+1})*N(y_{n+1}-y_{n+2},\frac{1}{n+1})* \\
&\qquad \cdots*N(y_{n+q-1}-y_{n+q},\frac{1}{n+1})
\end{aligned}$$

$$\begin{aligned}
&\geq N(y_n-y_{n+1},\frac{1}{n+1})*\cdots \\
&\quad *N(y_{n+q-1}-y_{n+q},\frac{1}{n+q} \\
&\geq (1-\frac{1}{n+1})*(1-\frac{1}{n+2})*\cdots*(1-\frac{1}{n+q}) \\
&\to 1 \text{ as } n \text{ tends to } \infty
\end{aligned}$$

Therefore $\{y_n\}$ is a Cauchy sequence in X. Since X is complete $\{y_n\}$ converges to y in X.

Next we prove $x_n+M\to y+M$ in X/M.

$$\begin{aligned}
N((x_n+M)-(y+M),t) &= N(x_n-y+M,t) \\
&= \sup((N(x_n-y+m,t)/m\in M) \\
&= \sup((N(x_n+m-y,t)/m\in M) \\
&\geq N(x_n+m-y,t)\quad \forall\, m\in M
\end{aligned}$$

Since $y_n = x_n + m_n$ for some $m_n \in M$, we have,

$$N((x_n + M) - (y + M), t) \geq N(y_n - y, t) > 1 - \epsilon, \quad \forall n \geq n_1,$$

implies $x_n + M \to y + M$.
Thus we have obtained a convergent subsequence of the Cauchy sequence $\{s_n + M\}$, this means $\{s_n + M\}$ is convergent.

Therefore X/M is complete. □

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