C 83624 (Pages : 3) Name....................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathematics

MT 2C 07—REAL ANALYSIS – II

Time : Three Hours Maximum : 36 Weightage

Part A

Short answer questions.
Answer **all** *questions.*
Each question has 1 weightage.

1. Let A E L ($\mathbb{R}$, ——) and B E L ($\mathbb{R}$, $\mathbb{R}$). Prove that $\|BA\|$ $\|B\|\,\|A\|$.

2. Let X and Y be vector spaces and let A E L (X, Y) be such that for all x E X Ax = 0 implies x = 0. Prove that A is one to one.

3. Let f $\mathbb{R}$ $\to R^1$ be given by $f(x, y, z) = x^3 + y^3 + z^3 + x^2 + y^2 + z^2$. Find the gradient of f at (2, 3, 1).

4. State inverse fu ion theorem.

5. Let $f = (f_1, f_2)$ ——ıe mapping of R^2 into R^2 given by f_l (x, y) = e^x cos y, $1_2 (x, y) = ex$ sin y. Show that the Jacobian of f is not zero at any point of R^2.

6. Let At be a a - algebra and let $\{E_i\}$ be a sequence of elements in A. Prove that

$$\bigcap_{i=1} E_i \; E$$

7. Prove that if $m^*(A) = 0$, then $m^*(A \text{ u } B) = m^*(B)$.

8. Let $\{E_i\}$ be a sequence of disjoint measurable sets and A be any set. Prove that :

$$m^*\left(A \cap \bigcup_{i=1} E_i\right. = m^* \; (A \cap E_i).$$

9. Is the characteristic function X (0,1) measurable ? Justify your answer.

Turn over

10. Let f and g be measurable functions defined on a set E of finite measure. If $f = g$ *a. e.*, then prove that $\int_E f = \int_E g$.

11. Let f be a measurable function. Prove that f^+ and f are measurable. Also prove that $f = f^+ - f^-$.

12. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ in measure. If $f_n \to f$ *a.e.* ? Justify your answer.

13. For functions f and g, prove that $\mathbf{D}_+ (f + g) \quad \mathbf{D}_+ f + \mathbf{D}_+ g$.

14. If f is absolutely continuous on [a, b] and if $f(x) \neq 0$ for all x E [a, b], then prove that 1 is absolutely continuous on [a, M.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *from the following ten questions.*
Each question has weightage 2.

15. Let $f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \dfrac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \end{cases}$ ~~and u be any unit vecto~~r in R^2. Show that the directional derivative $(D_u f)(0, 0)$ exists.

16. Let $[A]_1$ be the matrix obtained from the matrix [A] by interchanging two columns. Prove that $\det [A]_1 = -\det [A]$.

17. Prove that the outer measure is translation invariant.

18. Let E be a measurable set and let E> 0. Prove that there is an open set 0 DE such that $m^*(O \quad E) < E$.

19. Let $E_1, E_2, \ldots, E_n$ be a disjoint collection of measurable sets and let $\varphi = \quad a_i m(E_i)$. If m $(E_s) < \infty$ for each i, then prove that $\int \varphi = \sum_{i=1} a_i m(E_i)$.

20. Let f [0, 1] I 1 ~~be given by $f(x) =$~~ $\frac{0 \text{ if } x \text{ is rational}}{n \text{ if } x \text{ is irrational}}$ where n is the number of zeros immediately after decimal point in the representation of x. Show that f is measurable and evaluate $\int_{[0,}$

21. Let $\{f_n\}$ be a sequence of non-negative measurable functions that converge to f and let $f_n \leq f$ for each n. Prove that $\int f = \lim \int f_n$.

22. Show that if f is integrable over a measurable set E, then I$\int f|$ When does equality occur ? Justify your answer.

23. If f is of bounded variation on [a, b], then prove that $f'(x)$ exists for almost all x in [a, b].

24. Prove that absolutely continuous functions on $[a, b]$ are of bounded variation on [a, M.

(7 x 2 = 14 weightage)

Part C

Answer any two from the following four questions.
Each question has weightage 4.

25. Let E be an open set and let f E $\to \mathbb{R}^m$ be a mapping differentiable at a point x E E. Prove that the partial derivatives (Di n) (x) exist and $f' (x) e_j = \sum_{=1} (D_j f_i)(x) u_i$ where 1 j n.

26. (i) Prove that there exists a non-measurable set.

(ii) Prove that Cantor set is of measure zero.

27. (i) Prove that for each a E Et, the interval (a, ∞) is measurable.

(ii) Let f and g be non-negative measurable functions defined on a measurable set E. Prove that $\int_E f + g = \int_E f \quad \int g$.

28. (i) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Prove that there is a subsequence $\{f_{n_k}$ that converges to f almost everywhere.

(ii) Let f be an integrable function on [a, b] and let F (x) = F (a) + $\int_a f(t)\, dt$. Prove that F' (x) = $f(x)$ for almost all x in $[a, b]$.

(2 x 4 = 8 weightage)