D 52710 (Pages : 3) Name...

Reg. No

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2014

(Non-CUCSS)

Mathematics

Paper XII—FUNCTIONAL ANALYSIS—I

Time : Three Hours Maximum : 80 Marks

Part A

Answer all *questions.*
Each question carries 4 *marks.*

I. (a) Show that the metric space L' *b])* is not separable.

(b) Let X be a normed space C_{00} with I I$_1$. Show that the linear functional f on X defined by

$f(x) = \sum_{J=1} \underline{x\ (J)}$— for x c X is continuous and determine VII •

(c) Let (x_n) be a sequence in a Hilbert space H. Show that if $\sum_{n=1} \| \cdot\cdot_n$ then $\sum_{n=1}$ converges in H.

(d) Let X denote the sequence space 1^2. Let I $\|$ be a complete norm on X such that if $\|x_n - x\| \to 0$, then x_n (J) x (J) for every J = 1, 2, ... Then show that $\| \ \|$ is equivalent to the usual norm $\| \ \|_2$ on X.

(4 x 4 = 16 marks)

Part B

Answer any four *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Show that for 1 $p < \infty$, the metric space l^p is separable.

(b) Let X be a compact metric space. Show that X is complete and totally bounded.

(c) Show that the set of all polynomials in one variable is dense in C ([a, b]) with the sup metric.

Turn over

III. (a) Show that for 15 $p \leq \infty$, the metric space If (E) is complete for any closed interval E [a, b].

(b) Let x $\pi]$). For $n = 0, \pm$ $\pm 2, \ldots,$ let $\hat{x}(n) = \frac{1}{}$ $\int_{-\pi} x(t) e$ $dm(t)$ • Show that the series

$n =$ $\frac{\underline{x}(n) - e(-n)}{}$ converges in R.

IV. (a) Let Y be a closed subspace of a normed space X. For x y in the quotient space X/Y, let

$+ Y =$ $\{x + y$ 1:yc Y$\}$.

Show that is a norm on X/Y.

(b) Let X be a normed space. Show that if E_1 is open in X and E2 c X, then $E_1 +$ E2 is open in X.

(c) Show that the closed unit ball in 1^2 is convex, closed and bounded, but not compact.

Unit II

V. (a) Let X and Y be normed spaces and F X $\rightarrow$ y be a linear map. Show that F is continuous at 0 iff F is continuous on X.

(b) Give an example of a discontinuous linear map on a normed space.

VI. (a) Let Y be a subspace of a normed space X over K. Show that if a c X but a 4 $\overline{Y}$, then there is some f s X' such that f $- 0$, $f($ $) =$ dist (a, Y) and $= 1$.

(b) Let X be a normed space. Show that for every Y of X and every g s Y', there is a unique Hahn-Banach extension of g to X iff X' is strictly convex.

VII. (a) Let X be an inner product space. Show that for x, y s X ;

$$|< x, y >|^2 \quad < x, x > < y, y >,$$

where equality holds iff the set {x, y} is linearly dependent.

(b) State and prove Bessel's inequality.

(c) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff span $\{u_\alpha\}$ is dense in H.

Unit III

VIII. (a) Let X be a normed space and Y be a closed subspace of X. Show that Xis a Banach space iff Y and X/Y are Banach spaces in the induced norm and the quotient norm respectively.

(b) Let X be a normed space and Y be a Banach space. Let X_0 be a dense subspace of X and F_0 c BL (X_0, Y). Show that there is a unique F c BL (X, Y) such that ^{F} X_0 = Fo and 11F11=11Foll•

IX. (a) Let X = {x c Ca-7c, π]) : $x(\pi) = x(-\pi)$} with the sup norm. Show that the Fourier series of every x in a dense subset of X diverges at 0.

(b) State and prove the bounded inverse theorem.

X. (a) Let X be a normed space and $f : X \rightarrow K$ be linear. Show that f is closed iff f is continuous.

(b) Let X and Y be Banach spaces and F : X -r y be a linear map which is closed and surjective. Show that F is continuous and open.

(c) Show that the result in part (b) may not hold for normed spaces.

(4 x 16 = 64 marks)