D 91595 **(Pages : 3)** **Name**

Reg. No.................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS—I

Time : Three Hours **Maximum : 36 Weightage**

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Prove or disprove : A sequence in a metric space is bounded in X iff it is Cauchy.

2. Show that the metric space r is complete.

3. Define n^{th} Dirichlet Kernel D_n and evaluate $\int D_n (t)\, dt$.

4. State Riesz's lemma.

5. Let Y be a subspace of a normed space X. Show that $Y^0 \neq \phi$ iff $Y = X$.

6. Illustrate with an example that a linear map on a linear space X may be continuous with respec to some norm on X, but discontinuous with respect to another norm on X.

7. Let X be an inner product space and $x \in X$. Show that $(x, y) = 0$ for all $y \in X$ iff $x = 0$.

8. Show that if E and F are closed subsets of a Hilbert space H and $E \perp F$, then $E + F$ is close(H.

9. Let X be an inner product space. Show that if $E \subset X$ is convex, then there exists at most one approximation from E to any $x \in X$.

10. Let X be a normed space over K. Let $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Sh there are $f_1, f_2, \quad f_m$ in x' such that $f_j(a_i) = \delta_{ij}, 1 \quad j \; m$.

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11. With usual notations, show that C_c (T) is not closed in C_o (T).

12. Define Schauder basis for a normed space X and show that if there is a Schauder basis for a normed space X, then X must be separable.

13. Show that the linear space C_{oo} cannot be a Banach space in any norm.

14. State Uniform boundedness principle and interpret it geometrically.

(14 x 1 = 14 weightage)

Part B

Answer **any seven** *questions.*
Each question carries 2 *weightage.*

15. Show that the set of all polynomials in one variable is dense in c $([a, b])$ with the sup metric.

16. Show that the metric space $LP\left([a, bp\right.$ is separable for 1 $p < co,$ but the metric space LO $([a, b])$ is not separable.

17. Show that for all x e K" ;

and

$\|x\|_1 \quad \sqrt{n}\,\|x\|_2 < n\,\|x\|_\infty$.

18. Let X = K^3 for $x = (x(1), x(2), x(3))$ e X, let $\|x\| = (\text{I}x\,(1)|^{-} + |x\,(2)|^{-})^{3/} + {}_{|}x\,(3)^{3^{-1/3}}$ **. Show that II II is a norm on K^3.**

19. **Let X and Y be normed spaces and Z be a closed subspace of X. Show that if F** E BL $(X/Z, Y)$ and we let F (x) = $(x + z)$ for x E X, then F E **BL (X,** Y) and 110 = $|$F

1. Show that if a non-zero Hilbert space **H over K has a countable** orthonormal **basis then H is linearly isometric to** K^n for some n, or to 1^2.

Let E be a non-empty closed convex sub-set of a Hilbert space H. Show for each x E H, there exists a unique best approximation from E to x.

22. Let $X = K^2$ with the norm II IL • Consider Y = {(x (1), x (2) E X) : x **(1)** = **x** (2)}, and define g E

by g *(x (1), x (2))* = x (1).

Show that the Hahn-Banach extensions of g to X are given by :

$f(x(1), x(2)) = t\, x(1) + (1 - t)\, x(2)$, where $t \in [0,1]$ is fixed.

23. Show that a normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X.

24. Show that a subset E of a normed space Xis bounded in X iff f(E) is bounded in K for every f E X'

(7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. Show that every finite dimensional subspace of a normed space X is closed in X.

26. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff space $\{u_\alpha\}$ is dense in H.

27. State and prove Hahn-Banach separation theorem.

28. Let X be a normed space and Y be a closed subspace of X. Show that X is a Banach space iff Y and X are Banach spaces in the induced norm and the quotient norm, respectively.

(2 x 4 = 8 weightage)