D 51670 (Pages : 3) Name.....................................

Reg. No.................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2013

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS—I

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *the questions.*
Each question carries ***weightage*** *1.*

I. Show that every Cauchy sequence in a metric space is bounded. Is the converse true ? Justify your answer.

2. Show that $\int^{\pi} |D_n(t)|\, dt \to \infty$ as n 00 where D_n denotes the nth Dirichlet Kernel.

3. Prove that the norm function on a linear space is uniformly continuous.

4. "Let X be a normed space. Show that if E_1 is open in X and $E_2 \subset X$, then $E_1 + E_2$ is open in X.

5. State Riasz Lemma.

6. Give an example of a discontinuous linear map from a normed space into a normed space.

7. Let X be a normed space and P E BL (X) satisfy $p^2 = p$. Show that II P II = 0 or II P II 1.

8. Show that among all the l^p – spaces, 1 p 00, only 1^2 is an innerproduct space.

9. Let X be an innerproduct space. Show that for x, y ε X, $\|x +$ yr =11x412 11y112 iff Re$(x,$ y) = 0.

10. Let F be a subspace of an inner product space X and x E X. Let y E F be such that x – y 1 F. Show that y is a best approximation from F to x.

11. Let X be a normed space over K and a ≠ 0 ε X. Show that there is some f ε X' such that f(a) = $\|a\|$ and 1.

12. Show that the linear space c_{00} cannot be a Banach space in any norm.

13. Is it true that every finite dimensional normed space is separable ? Justify your answer.

14. State uniform boundedness

(14 x 1 = 14 weightage)

Turn over

Part B

Answer **any seven** *questions.*

Each question carries weightage 2.

15. Show that the set of all polynomials in one variable is dense in C ——b]) with the sup metric.

16. Show that $L^{\infty}([a,b])$ is not separable.

17. Let x c L' $([-\pi,\pi])$. Show that $\hat{x}(n) \to 0$ as n $\to \pm$ 00, where 1(n) denotes the nth Fourier coefficient of x.

18. Show that the three norms II $\|_1$ II $\|_2$ and II $\|_\infty$ on K are equivalent.

19. Let X and Y be normed spaces and Z be a closed subspace of X. Show that if F E BL $\left(X/Z, Y\right)$ and if we let $F(x) = \tilde{F}(x+Z)$ for x E X, then F ε BL (X, Y) and $\|F\| = \|\tilde{F}\|$.

20. Let X and Y be inner product spaces. Show that a linear map F: X $\to$ Y satisfies $\langle F(x), F(y)\rangle = (x, y)$ for all x, y E X iff it satisfies $\|F(41$ $\|x\|$ for all x c X, where the norms on X and Y are induced by the respective inner products.

21. Let E be a non-empty closed convex subset of a Hilbert space H. Show that for each $x E$ H, there exists a unique best approximation from E to x.

22. Let X be a normed space over K, Y be a subspace of X and g E Y. Show that there is some f c X' such that $f/_Y = g$ and $\|f\| = \|g\|$.

23. Show that a normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X.

24. Let X and Y be Banach spaces and F_n E BL (X, Y), n =1, 2, ... Show that there is some F ε BL (X, Y) such that F_n (x) $\to F(x)$ for every x ε X iff (F_n (x)) converges for every x in some set E whose span is dense in X and the set $\{\|F_n\| : n = 1, 2, \ldots\}$ is bounded.

(7 x 2 = 14 weightage)

Part C

Answer any two questions.

Each question carries weightage 4.

25. Show that for 1 $p <$ a, the metric space JP is complete.

26. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha$ is an orthonormal basis for H iff x E H and $(x, u_\alpha\rangle = 0$ for all a, then x = 0.

27. Let X be a normed space over K, E be a non-empty open convex subset of X and Y be a subspace of X such that E $\cap$ Y = ϕ. Show that there is a closed hyperspace Z in X such that Y c Z and E *n Z* = 4).

28. Let X be a normed space and Y be a closed subspace of X. Show that X is a Banach space iff Y and X/Y are Banach spaces in the induced norm and the quotient norm, respectively.

(2 x 4 = 8 weightage)