C 3544 (Pages : 3) Name.......................................

Reg. No...................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CUCSS)

Mathematics

MT 4C 15—FUNCTIONAL ANALYSIS – II

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Show that if F is a bijective closed map on a normed space then F^{-1} is also a closed map.

2. Let X be a Banach space and A e $BL(X)$. Show that if k ε a, (A), then Ile 1511 A $_{\text{II}}$.

3. Let X be a normed space and A s $BL(X)$ be invertible. Show that $\lim\limits_{n\to\infty} \inf_{\text{II}} A^n \|^{1/n} > 0$

4. Let Y be a subspace of a normed space X. For x ε X , let $F(x$ x/Y. Show that F is a surjective linear map from X to Y' such that II $F(x)$ II 5 II x II for all x e X'•

5. Let X be a Banach space. Show that if X is reflexive then it remains reflexive in any equivalent norm.

6. Let. M = $\operatorname{diag}(k_1, k_2, \ldots)$ and X be the sequence space . Show that if $k_{,1}$ --> 0 as n ---> co , then p. defines a map in CL (X).

7. State Riesz representation theorem.

8. Let H be a Hilbert space and (x_n) be a sequence in H. Show that x_n $\rightharpoonup$ in H if $\langle x_{n,y}\rangle \to \langle x, y$ uniformly for y E H with II y II 5.1 .

9. Let X be a non-zero Banach space and P ε $BL(X)$ be a projection. Show that if 0 P ——th $\sigma(P) = \{0,1\}$.

Turn c

10. Let H be the Hilbert space C^2 and $A : H \quad H$ be defined by :

$A (x(1), x(2)) (2x (1), x (2))$ for $(x(1), x(2))$ s H. Determine A*.

11. Let H be a Hilbert space. Show that if (A_n) is a sequence of unitary operators on H, and A ε BL(H) be such that II $A_n - A$ 0 as n ∞, then A is unitary.

12. Let H be a Hilbert space and A c BL(H) be self-adjoint. Show that A^2 0 and A <II A H I.

13. Show that if x_1 and x_2 are eigen vectors of a normal operator on a Hilbert space H corresponding to distinct eigen values, then x1 I x2 •

14. Let **H** be a non-zero Hilbert space and **A** E BL(H). Show that :

$$\text{II } A \| = \sup\{\lceil k : k \ \varepsilon \ a \ (A \quad A)\}$$

(14 x **1** = **14** weightage)

Part B

Answer **any seven** *questions.*
Each question carries 2 *weightage.*

15. Let X and Y be Banach spaces and F s BL(X,Y). Show that **R (F)** is linearly homeomorphic to $X/Z(F)$ iff R (F) is closed in Y.

16. Let X be a Banach space and A c BL(X). Show that A is invertible iff A is bounded below and the range of A is dense in X.

17. Let X be the sequence space 1^2 and $A : X \quad X$ be defined by :

$A (x) = \left(0, x(1), \frac{x(2)}{2}, \frac{x(3)}{3}, \ldots\right)$ **for x** ε **X**. Show that $\sigma_\varepsilon (A) = \phi$ and σ_u **(A)** = **{0}** = $\sigma(A)$.

. Let X and Y be normed spaces and F c BL(X,Y). Show that II F ‖ = ‖ F 11=11F ‖ and $F'' J_X = J_Y F$, where h and J_Y are the Canonical embedding of X and Y into X" and Y" respectively.

19. Let X be a reflexive normed space show that X is separable iff X is separable.

20. Let X and Y be Banach spaces and F : X Y be linear. Show that F is continuous and of finite rank iff F is a compact map and R (F) is closed in Y.

21. Let H be a Hilbert space and $f \in H$. Show that if g $g \| = \| f \|$ and $g(x) = 1(x)$ for some non-zero $x \in Z(f)$, then $g = f$.

22. Let H be a Hilbert space and $A \in BL(H)$ be self-adjoint. Show that

$$\| A \| = \sup\{|< A(x), x >| : x \in H, \| x \| \leq 1\}.$$

23. Let H be a Hilbert space and $A \in BL(H)$. Show that σ_n (A) $\subset w(A)$ and $\sigma(A)$ is contained in the closure of $w(A)$.

24. Let H be a Hilbert space and $A \in BL(H)$. Show that A is compact iff $A^* A$ is compact.

(7 x 2 = **14** weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. State and prove closed graph theorem.

26. Let F be a finite dimensional subspace of a Hilbert space H. Show that $H = F + F^{\perp}$ and $F^{\perp\perp} = F$.

27. Let H be a Hilbert space and $A \in BL(H)$. Show that R (A) = H iff A^* is bounded below, and R (A^*) = H iff A is bounded below.

28. Let A be a non-zero compact self adjoint operator on a Hilbert space **H** over K. Show that there exist a finite or infinite sequence (S_n) of non-zero real numbers with $|S_1| \geq |S_2| \geq |S_3| \geq \ldots$ and an orthonormal set $\{u_1, u_2, \ldots\}$ in H. Such that $A(x) = \sum S_n < x, u_n > u_n$, $x \in H$.

(2 x 4. = 8 weightage)