C 42502 (Pages : 3) Name..

Reg. No.................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2013

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum 36 Weightage

Part A

Answer all *questions.*
Each question carries 1 weightage.

1. For the given function *f*, sketch level sets $f^{-1}(c)$ at the heights indicated.

   $f \quad x_2) = \quad - \quad c = -1, 0, 1.$

2. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1^2 +$

3. Show by example that the set of vectors tangent at a point p of a level set might be all of $\mathbb{R}p^{+}$.

4. Sketch the surface of revolution obtained by rotating c about the x_1 axis where c is the curve $- \, ^2 + ^2 = 1, x_2 > 0.$

5. Show that the plane R^2 is connected.

6. Describe the spherical image of the cone.

   $- \, ^2 + x_2^2 \quad x_3^2 = 0, x_1 > 0.$

   (the surface is oriented by where f is the function $- x_1^2 + x_2 + x_3$).

7. Define a geodesic on an n-surface $S \subset \mathbb{R}^{+1}$. If S contains a straight line segment, prove that segment is a geodesic.

8. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $a : I \to S$ be a parametrised curve is S ; X, Y smooth vector fields tangent to S along a. Then prove that (X . $= X' \cdot Y + X \cdot Y'$

Turn over

9. Compute the derivative ∇_{v} (X) where v E $\mathbb{R}^{\hat{}}_{p}$, p E R^2 given by X $(x1, x2$ Xi, X2 – X2, Xi),

v = (cos 0, sin 0 – sin 0, cos 0).

10. Find global parametrisation of x_2 – = 0. (You may choose the orientation).

11. Find the length of the parametrised curve a : I → R^4 where a (t) = (cos t, sin t, cos t, sin t),

I = [0, 2π].

12. Let S be an oriented n - surface in $\mathbb{R}^{n+1}$ Define the first and second fundamental forms of S at p E S.

13. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let p E S. Give a formula for computing K(p), the Gauss-Kronecker curvature.

14. Let U be an open set in $\mathbb{R}^n$ and let Q : U → $\mathbb{R}$ be a smooth map. Define : dQ, the differential of Q.

(14 x 1 = 14 weightage)

Part B

Answer any seven *questions.*
Each question carries 2 *weightage.*

15. Let X be the vector field on R^2, X (p) = $(p$, X $(p))$ where X $(x_1, x_2) = \left(-2x_2, \frac{1}{2}\right)$. Find the integral curve of X through $p = (1, 1)$.

16. Show that the maximum and minimum values of the function $g(x_1,$ $= \sum_{i=1}^{n+1} a_{ij}\, x_i\, x_j$ on the unit n-sphere $\sum_{i=1}^{n+1} xi^2 = 1$, where (ad) is a symmetric n real matrix, are eigen values of the matrix (a_y).

17. Show that the two orientations on the n-sphere $\sum_{i=1}^{n+1} x_i^2 = r^2$ of radius $r > 0$ are given by

N_1 (p) = (p, p 1 r) and **N2** (p)= (p, – p I r).

18. Describe the spherical image of the parabola – x_1 + x2 = 0 (orientation left to your choice).

• 19. Let S denote the cylinder x_1 + x2 =1 in $\mathbb{R}^-$ Show that a is a geodesic of S iff a is of the form

a (t) = (cos $(at + b)$, sin $(at + b)$, [Ct + Cd)), for some a, b, c, d E R.

20. Let a : [0, π] s^2 be the half great circle in s^2 running from p = (0, 0, **1) to** q = (0, 0, – **1)** defined by a (t) = (sin t, 0, cos t). Let v = (p, 1, 0, 0) E **S** p, show that P_a (v) = (q,— 0, 0).

21. Choosing your own orientation, compute the Weingarten map for the circular cylinder $x_2^2 + \hat{x_3} = 1$ in R^3

22. Let C be a connected oriented plane curve and let 13 : I → C be a unit speed global parametrization of C. Then prove that 13 is periodic if and only if C is compact.

23. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+}$. Then, prove that the Gauss - **Kronecker** curvature of S at p is non - zero for all p E S if and only if second fundamental form s_p of S at p is definite for all p E S.

24. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let

$\pi(S) = \{v \in \mathbb{R}_p^{n+1} \subseteq \mathbb{R}^{2n+2}$: p E S and v • N (p) Prove that T (S) is a 2n— surface in $\mathbb{R}^{n+2}$.

(7 x 2 = 14 **weightage)**

Part C

Answer any **two** *questions.*
Each question carries 4 ***weightage.***

25. Let U be an open set in $\mathbb{R}^n$ +1 and let f = U -3111 be smooth. Let p E U be a regular point off and let $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to [V $f(p)$]
(Both set inclusion to be proved)

26. Let S be an n-surface in $\mathbb{R}^{+1}$, let p E S and let v E S. Then prove the existence and 'uniqueness' of the maximal geodesic in S passing through p with initial velocity v.

27. Prove that the Weingarten map **L*p*** is **self-adjoint.**

28. Let S be a compact oriented n-surface in $\mathbb{R}^n$ +1 Prove there exists a point p on S such that the second fundamental form at p is definite.

(2 x 4 = 8 **weightage)**