

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MAY 2014

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours

Maximum : 36 Weightage

Part A*Answer all questions.**Each question carries 1 weightage.*

1. Show that the graph of any function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a level set for some function $F : \mathbb{R}^n \rightarrow \mathbb{R}$.
2. Sketch the vector field X ($X(p) = \mathbf{P}, X(p)$) where $X(p) = -p$.
3. Sketch the level set $f^{-1}(0)$ and typical values of the vector field ∇f for $p = f^{-1}(0)$ when $f(x_1, x_2) = x_1^2 - x_2 - 1$.
4. Let S be an $(n-1)$ -surface in \mathbb{R}^n given by $S = f^{-1}(C)$ where $f : U \rightarrow \mathbb{R}$ (U open in \mathbb{R}^n is such that $\nabla f(p) \neq 0$ for all $p \in S$). Define the cylinder over S in \mathbb{R}^n and show that it is an n -surface in \mathbb{R}^n .
5. Describe the spherical image of one sheet of 2-sheeted hyperboloid $x^2 - x_2^2 - x_3^2 = 4, x_1 > 0$, oriented by $N = \frac{\nabla f}{\|\nabla f\|}$ where $\mathbf{f}(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2$.
6. **Prove** that geodesics have constant speed.
7. Let X and Y be smooth vector fields along the parametrized curve $\alpha : I \rightarrow \mathbb{R}^{n+1}$. Verify that $[X \cdot Y]' = Y \cdot X'$.
8. Define the Weingarten map $L_\nu : S_p \rightarrow S_p$ (with standard notation).
9. Compute $\nabla_\nu X$ where $\nu \in \mathbb{R}^3, p \in \mathbb{R}^2$ and X is given by $X(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2), \nu = (1, 0, 0, 1)$.
10. Let C be an oriented plane curve and let $p \in C$. Define γ : a parametrization of a segment of C containing p .

Turn over

11. Find the length of the parametrized curve $\alpha: I \rightarrow \mathbb{R}^3$ where $I = [-1, 1]$ and $\alpha(t) = ((\cos 3t, \sin 3t, 4t))$.
12. Let S be an oriented n -surface in \mathbb{R}^{n+1} and let $p \in S$. Define the first and second fundamental forms of S at p .
13. Show that a parametrized 1-surface is simply a regular parametrised curve.
14. Let $\phi: V_1 \rightarrow V_2$ and $\psi: V_2 \rightarrow \mathbb{R}^k$ be smooth verify the chain rule $d(\psi \circ \phi) = d\psi \circ d\phi$.

(14 x 1 = 14 weightage)

Part B

*Answer any seven questions.**Each question carries 2 weightage.*

15. Find the integral curve through $p = (a, b)$ of the vector field X given in question 2.
16. Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Determine the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$.
17. Let $S \subset \mathbb{R}^{n+1}$ be an oriented n -surface. Prove that there exists on S exactly two smooth normal vector fields.
18. Choosing an orientation, describe the spherical image of the cylinder $\sum_{i=2}^{n+1} x_i^2 = 1$.
19. Show that a parametrized curve α in the unit sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ is a geodesic iff (if and only if) it is of the form

$$\alpha(t) = (\cos at) e_1 + (\sin at) e_2$$

for some orthogonal pair of unit vectors $\{e_1, e_2\}$ in \mathbb{R}^{n+1} and some $a \in \mathbb{R}$

20. Let S be the n -sphere $\sum_{i=1}^{n+1} x_i^2 = r^2$ oriented by the inward unit normal vector field. Prove that the

Weingarten map of S is multiplication by $-\frac{1}{r}$ ($r > 0$)

21. "Local parametrization of plane curves are in principle, easy to obtain". Explain the statement and illustrate with an example.

22. Let C be a connected oriented plane curve and let $\beta: I \rightarrow C$ be a unit speed global parametrization of C . Prove that β is either one-to-one or periodic.

23. Find the Gaussian curvature of the Cone

$$x_1^2 + x_2^2 - x_3^2 = 0, x_3 > 0.$$

24. State and prove the inverse function theorem for n -surfaces.

(7 x 2 = 14 weightage)

Part C

Answer any two questions.

Each question carries 4 weightage.

25. Let U be an open set in \mathbb{R}^n and let $f: U \rightarrow \mathbb{R}^m$ be smooth. Let $p \in U$ be a regular point of f and let $C = f^{-1}(f(p))$. Then prove the set of all vectors tangent to $f^{-1}(f(p))$ at p is equal to $[\ker Df(p)]^\perp$ (Both set inclusion to be proved).

26. Let S be a compact, connected oriented n -surface in \mathbb{R}^n . Prove that the Gauss map maps S onto the unit sphere S^{n-1} .

27. Let C be a connected, oriented plane curve. Prove : there exists a global parametrization of C .

28. "Locally n -surfaces and parametrized n -surfaces are the same". State the theorems which lead to the above assertion and outline their proofs.

(2 x 4 = 8 weightage)