SOME CONVEXITY PROPERTIES OF NORMED SPACES.

MINOR RESEARCH PROJECT

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INTRODUCTION

1.1 Definition and Preliminaries

The Concept of Convexity is a very old topic which is very simple and natural notion. Motivated by the properties of convex bodies such as the five platonic solids and other polyhedra, and their structures it was systematically studied by Newton Minkowski and Others. The properties of convex sets are classified mainly in to three aspects; qualitative, quantitative and combinatorial.

It is well known that, a subset C of a real vector space is convex if and only if it contains with any pair of points in C the entire line segment joining them. It can be easily observed that intersection of any family of convex sets is convex, even though the intersection may be empty. T he famous Helly type theorems made tremendous impact in the development of Combinatorial convexity theory and has been studied, applied and generalized by many others. These theorems are stated as follows.[2]

Helly's Theorem: Let B= { B_1, B_2…B_m} be acollection of convex sets in R^n with $m \geq n+1$. If every subfamily of n+1 sets in has nonempty intersection, then $\bigcap_{i=1}^{m} B_i$ is nonempty.

Caratheodory's Theorem: If S is a nonempty subset of then every x in the convex hull of S can be expressed as a convex combination of n+1 or fewer points.

Radon's Theorem: Let S = { x_1, x_2 …….x_m} be any finite set of points in R^n. If m ≥ n+2, then S can be partitioned in to two disjoint sets S_1 and S_2 such that$Co(S_1) \cap Co(S_2)$ is nonempty.

Not only to generalize these classical theorems, but also to unify properties of a variety of Mathematical structures such as vector spaces, lattices, metric spaces and graphs, an axiomatic foundation of convexity was laid down by Levi.

Definition: A convexity on a set X is a collection C of subsets of X such that i). The empty set φ and X are in C .

ii). C is stable for intersection, that is, If $D \subset C$ is nonempty, then $\bigcap\{ C \subset D \}$ is in C.

iii). C is stable for nested Union,that is, If $D \subset C$ is nonempty and totally ordered under inclusion, then $\bigcup\{ C \subset D \}$ is in C.

C is called a convexity on X , (X,C,) is called convexity space or convex structure and members of are called convex sets. For any set A ⊂ X, Co(A) = $\bigcap\{ C \subset X : A \subset C \in C\}$

Is called the convex hull of A. Convex hull of a finite number of points is called apolytope.

Example:1).Let X = R and let C= { A⊂R: if x, y ∈ A and x < z < y , then,z ∈ A.} Then C. is a convexity on R

Example:2).For any nonempty set X, {φ, X} and P(X) are clearly convexity spaces.

Example:3). Let X = R^n, and let C= {A⊂R^n: if x, y ∈A and 0 < t < 1, then,x+(1-t) (y-x) ∈ A}. Then C. is a convexity on R^2

Definition: Let (X,**C,**) be a convexity space, C ∈ **C** is said to be a half space if its compliment is convex.

Definition: A subset A in a convexity space X is convexly independent if for any point $x \ in\ A, x \notin Co\ (A - x)$

Example 4): Any two distinct elements in R are convexly independent and any three distinct elements are convexly dependent. In R^2 any three non collinear points are convexly independent. All the points on a Circle are convexly independent..

Example 5): In R^2 ,with respect to usual convexity, The convex hull of a finite set A is the polygonal area determined by a maximal convexly independent subset of A.

Theorem 1): Let X be a Real normed space and f be a bounded linear functional on X and a $\alpha \in R$. Then the sets $\boldsymbol{E_1} = \{ \in X\colon\ f(x) > \alpha\}$,$\boldsymbol{E_2} = \{ x \in X\colon\ f(x) \geq \alpha\}$, $\boldsymbol{E_3} = \{ x \in X\colon\ f(x) < \alpha\}$,and $\boldsymbol{E_4} = \{x \in X\colon\ f(x) \leq \alpha\}$ are half spaces

Proof: We prove $\boldsymbol{E_1}$ is a convex set. The convexity of the other sets may be proved in a similar manner. Then each is a half space follows from the fact that $\boldsymbol{E_1}$ is the compliment of $\boldsymbol{E_4}$ and that $\boldsymbol{E_2}$ is the compliment of $\boldsymbol{E_3}$**.**

Let $x, y \in \boldsymbol{E_1}$, and $t \in (0,1)$Then,

$\big(x + (1-t)(y-x)\big) = f(x) + (1-t)\big(f(y) - f(x)\big) > \alpha$.

That is, $x + (1-t)(y-x) \in$ C Hence C is convex

■

Example: 4). Let X = R^n and the convexity be defined as in example 3 and f be a linear functional on X. Let $\alpha \in$ R and C = { x$\in$ X : $f(x) \leq \alpha$ }. Then C is convex, for, if x,y$\in$C and t$\in$ (0,1). Then, $\big(x + (1-t)(y-x)\big) = f(x) + (1-t)\big(f(y) - f(x)\big) \leq \alpha$.

That is, $x + (1-t)(y-x) \in$ C Hence C is convex. The compliment of C is the set C^c ={ x $\in$ X : $f(x) > \alpha$ } is also convex. Hence C is a half space.

Definition: Let (X, C) be a convexity space. C is said to be of *arity* $\leq$ n if its convex sets are determined by polytopes. That is, a set C is convex if and only if Co(F) $\subset$ C for each subset F of cardinality at most n.

Example 5): The Euclidean convexity on R^n is of arity 2 because it is induced by an interval function defined by

$$I(x, y). = \{ tx + (1-t)y : t \in (0, 1) \}$$

Example 6): Let X be any set with more than five elements. Let $\mathcal{C}$ be the collection of subsets

C of X such that $|C| \leq 5$ and X. Then , is a convexity on X and it is of arity 6.

1.2. Seperation Axioms

Definition: A convexity space X is said to have the ***separation property***

S_1 : If all singletons are convex.

S_2 : If any two distinct points are separated by half spaces. That is,

for any $x_1, x_2 \in X, x_1 \neq x_2$, there is a half space H of X such that

$x_1 \in H$ and $x_2 \notin H$.

S_3 : If any convex set C and any point not in C can be separated by half spaces.

That is, if $C \subset X$ and $x \in X \setminus C$, then there is a half space H of X such that

$C \subset H$ and $x \notin H$.

S_4 : If any two disjoint convex sets can be separated by half spaces. That is,

if C_1 and C_2 are disjoint convex subsets of X, then there is a half space H

such that $C_1 \subset H$ and $C_2 \subset X \setminus H$.

The usual convexity on any vector space satisfies all the separation properties.

The trivial convexity consisting of the set X (with at least two elements) satisfies none of the above axioms.

Definition: A subset S of an interval space is star shaped at a point $p \in S$ if for every $x \in S$,

$I(x, p) \subset S$. The star centre is the set of all points at which S is star shaped.

Example7:

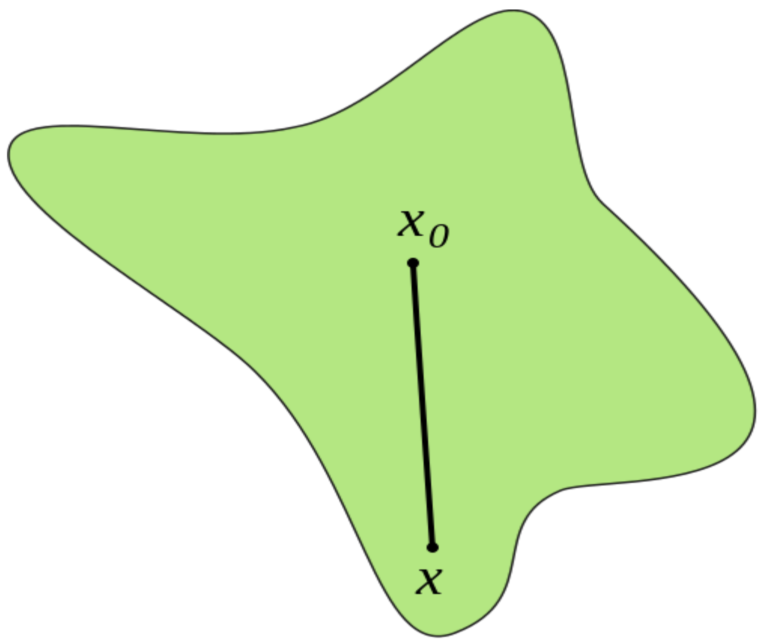


Figure 1

Set star shaped at x_o

The object in Figure 2. Gives en example of a set whose star centre is empty

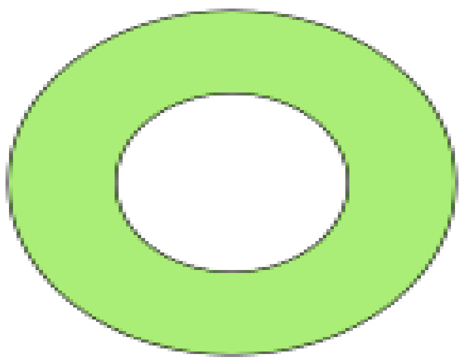


Figure 2.

Definition: Let X be a convexity space. Then,

1. The Helly number of X **h** is the smallest number n such that for each finite set $F \subset X$ with cardinality at least n+1 ,

$$\cap \{ Co \{F \setminus \{a\}: a \in F\} \neq \phi$$

(That is F is Helly (H-) dependent).

2. The Caratheodory number of X **c** is the smallest n such that for each $F \subset X$ with cardinality at least n+1,

   $Co(F) \subset \cup \{ Co(F \setminus \{a\}) : a \in F\}$

   (That is F is caratheodory (C-) dependent).

3. The Radon number of X **r** is the smallest number n such that any $F \subset X$ with cardinality at least n+1 can be partitioned in to sets F_1 and F_2 such that

   $Co(F_1) \cap Co(F_2) \neq \phi$

   (That is F is Radon (R-) dependent).

4. The exchange number (or Sierskma number) of X **e** is the smallest number n such that for each $F \subset X$ with cardinality at least n+1 and for each $p \in F$,

   $Co(F \setminus \{p\}) \subset \cup \{ Co(F \setminus \{a\}) : a \in F, a \neq p\}$.

   (That is F is Exchange (E-) dependent).

Attempts were made to find inter relation between these invariants and resulted in the following theorems.

Theorem: Levi's Theorem[]. Let (X, C) be a convexity space. Then the existence of **r** implies the existence of **h** and **h ≤ r.**

Theorem: Eckhoff- Jamison inequality []. If **c** and **h** exists for a convexity space, the r exists and

$\mathbf{r \leq c(h-1)}$ if $\mathbf{h} \neq 1$ and $\mathbf{c} < \infty$

Theorem: $\mathbf{e}-1 \leq c \leq \max \{h, e-1\}$

For the usual convexity on R^n, $h = c = r = e = n + 1$

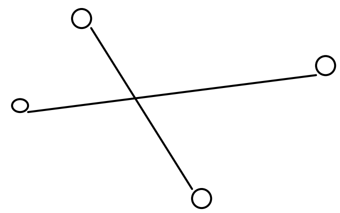


Figure 3.

Radon Partition

Definition: A convexity space is said to be join hull commutative if for any convex set C and any $p \in X$, $Co (C \cup \{p\}) = \cup \{ Co (\{c, p\}) : c \in C\}$.

Example 8): Let X be any set with more than six elements. Let $\mathcal{C}$ be the collection of subsets C of X such that $|C| \leq 5$ and X. Then , is a convexity on X. Let C be any sub set of cardinality 5 and p an element of X which is not in C Then $C \cup \{p\}$ is not convex and $Co(C \cup \{p\}) = X$. But

$$\cup \{ Co (\{c, p\}) : c \in C \} = C \cup \{p\}.$$

Gist of The Report

In this report we analyze different convex structures on normed spaces. In a normed space there is the convexity associated with the linear space structure and also convexity induced by the interval function defined by the norm. It is found that there is difference between the two. Also The H convexity symmetrically generated by a family of linear functional is also studied. It is also observed that in linear spaces of finite dimension greater than 2, there are convexity of infinite arity. An example is given to show that the H- Convexity need not be join hull commutative. Also the study of the parameters such as Helly number, Caratheodory number, Radon number and has been done.

CONVEX STRUCTURES IN NORMED SPACES

2.1. Normed spaces

Definition: Let X be a Linear space over a field K. A set $E \subset X$ is convex if f if

$$tx + (1-t)y \in E \text{ whenever } x, y \in E \text{ and } t \in (0,1).$$

Definition [1]: Let **X** be a Linear space. A norm on X is a Function

$\|.\| : \mathbf{X} \to$ R, satisfying the following properties.

For all x, y, z in **X** and k in the field K.

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if x = 0. (Positive definiteness)
2. $\|x+y\| \leq \|x\| + \|y\|$. (Triangle inequality)
3. $\|kx\| = |k| \, \|x\|$

Definition [1]: Let **X** be a Linear space over a field K. An inner product on X is a Function

$\langle . \rangle : \mathbf{XxX} \to$ K, satisfying the following properties.

For any x, y, z $\in$ X and k $\in$ K

i). $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if x = 0

ii). $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and

$\langle kx, y\rangle = k\,\langle x, y\rangle$

iii). $\langle x, y\rangle = \overline{\langle y, x\rangle}$

A linear space together with an inner product defined on it is called an inner product space.

Remark: An inner product defines a norm on the linear space X by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

Theorem 2.1(Cauchy Schwartz inequality): Let X be an inner product space and $\|.\|$ the norm induced by the inner product. Then for x, y $\in$ X,

$|\langle x, y\rangle| \leq \|x\| \ \|y\|$. And the equality holds if and only if x and y are linearly dependent.

■

Theorem 2.2): If the norm is induced by an inner product, the equality in the Triangle inequality holds only if y = kx for some $k \geq 0$. We have that,

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle$$

$$(\|x\| + \|y\|)^2 = \|x\|^2 + \|y\|^2 + 2\,\|x\|\ \|y\|.$$

Suppose equality holds in Triangle inequality, Then, $Re\langle x, y\rangle = \|x\| \ \|y\|$ which imply that $\langle x, y\rangle = \|x\| \ \|y\|$. Then by Cauchy Schwartz inequality, x and y are linearly dependent. Let y = kx. Then

$$\langle x, y\rangle = \langle x, kx\rangle = \bar{k}\,\langle x, y\rangle = \|x\| \ \|kx\| = |k|\ \|x\| \ \|y\|.$$ Similarly,

$$\langle y, x\rangle = \langle kx, x\rangle = k\,\langle x, y\rangle = \|kx\| \ \|x\| = |k|\ \|x\| \ \|y\|.$$

If $k \neq 0$ Then $k = |k| \geq 0$

■

2.2. Interval Convexity

Definition: Let X be any set. An interval function on X is a function $I : X \times X \to P(X)$ Such that

1. $I\,(a, b) = I(b, a)$
2. $a, b \in I(a, b)$

A Convexity is called **interval convexity** if its convexity is induced by an interval function. That is, $A \subset X$ is convex if $I\,(a, b) \subset A$ whenever $a, b \in A$.

Example: Let X be a normed linear space.Define $I: X \times X \to P(X)$ as,

$$I(x, y) = Seg\,(x, y) = \{z \in E : d(x, y) = d(x, z) + d(z, y).\}$$

Then, I defines an interval function on X and hence a convexity. This is different from the usual definition of convex sets in linear spaces.

Remark: In R^n, we usually define a set to be convex, if $tx + (1-t)y \in E$ whenever x, y $\in$ E and t $\in$ (0, 1).When R^n is given the Euclidean norm, $E \subset R^n$, x, y $\in$ E and t $\in$ (0, 1). Then,

$\|x-(tx+(1-t)y)\| = \|(1-t)x-(1-t)y\|$ = (1-t) $\|x-y)\|$

Similarly, $\|(tx+(1-t)y)-y\|$ = t $\|x-y\|$

Hence, $\|x-y\|$ = $\|x-(tx+(1-t)y)\|$ + $\|(tx+(1-t)y)-y\|$

Conversely if $\|x-y\|$ = $\|x-z\|$ + $\|z-y\|$

Then, $x-z = k(z-y)$ for some $k \geq 0$

That is, $z = \frac{1}{1+k}x + \frac{k}{1+k}y$. Put $= \frac{1}{1+k}$. Then $t \in (0,1)$ and z = tx + (1-t)y

Example: Let R^n be given the norms $\|.\|_1$, $\|.\|_\infty$ and the Euclidean norm ($\|.\|_2$)are defined by,

For $x = (x(1), x(2), \ldots\ldots x(n))$,

$$\|x\|_1 = \sum_1^n |x(i)|$$

$$\|x\|_\infty = Max\{\,|x(1)|, |x(12)|, \ldots\ldots\ldots |x(n)|\}$$ and

$$\|x\|_2 = \sqrt{\sum_1^n |x(i)|^2}$$

Let $n = 2$, $x = (1,1), y = ((-1,-1)$

Then the segment $seg(x,y)$ with respect to $\|x\|_2$ is the line segment joining x and y. But when we consider $\|x\|_1$

$$seg(x,y) = \{\, z = \big(z(1), z(2)\big) \colon \|x-y\|_1 = \|x-z\|_1 + \|z-y\|_1\}$$

$$= \{\, z = \big(z(1), z(2)\big) \colon |x(1)-y(1)| + |x(2)-y(2)| = |x(1)-z(1)| + |x(2)-z(2)| + |z(1)-y(1)| + |z(2)-y(2)|\}.$$

$$= \{\, z = \big(z(1), z(2)\big) \colon 4 = |1-z(1)| + |1-z(2)| + |z(1)+1| + |z(2)+1|\}.$$

$= \{ z = (z(1), z(2)) : -1 \leq z(1) \leq 1, -1 \leq z(1) \leq 1\}$

Remark: If x = (x(1), x(2)) and y = (y(1), y(2)),

Co (x, y) = { z = (z(1), z(2)) : z(1) is between x(1) and y(1), z(2) is between x(2) and y(2) }

Remark : When we consider $\|x\|_\infty$, x = (0,0) , y = (1,1). Then ,

$$seg(x, y) = \{ z = (z(1), z(2)) : \|x - y\|_\infty = \|x - z\|_\infty + \|z - y\|_\infty$$

$\|x - y\|_\infty$ = Max { $|x(1) - y(1)|, |x(2) - y(2)|$ }= 1

Now,For any $z = (z(1), z(2))$,

1 = $|x(1) - y(1)| \leq |x(1) - z(1)| + |z(1) - y(1)|$

$\leq$ Max { $|x(1) - z(1)|, |z(2) - y(2)|$ } +

Max { $|x(1) - z(1)|, |z(2) - y(2)|$ }

1 = $|x(2) - y(2)| \leq |x(2) - z(2)| + |z(2) - y(2)|$

$\leq$ Max { $|x(1) - z(1)|, |z(2) - y(2)|$ } +

Max { $|x(1) - z(1)|, |z(2) - y(2)|$ }

Hence, 1 $\leq$ Max { $|x(1) - z(1)|, |z(2) - y(2)|$ } +

Max { $|x(1) - z(1)|, |z(2) - y(2)|$ }

Thus,

$$seg(x, y) = \{ z = (z(1), z(2)) : Max\{ |x(1) - z(1)|, |z(2) - y(2)| \} + Max\{ |x(1) - z(1)|, |z(2) - y(2)| = 1\}$$

$= \{ z = (z(1), z(2)) : \{ |x(1) - z(1)| + \{ |x(1) - z(1)| =$ 2,
$|z(2) - y(2)| + |z(2) - y(2)| = 1\}$ is the line segment joining x and y.

Now we prove the following Theorem.

Theorem 2.3: Let $1 < p < \infty$ and $f: R^2 \to R$ be defined as,

$f(x,y) = (x^p + y^p)^{\frac{1}{p}} + ((1-x)^p + (1-y)^p)^{\frac{1}{p}}$, $0 \le x, y \le 1$. Then f is a constant if and only if x = y

Proof : First suppose that **x = y.**

Then, $f(x,y) = 2^{\frac{1}{p}}(x + 1 - x) = 2^{\frac{1}{p}}$, a constant.

Conversely suppose that f is a constant. Then $\frac{\partial f}{\partial x} = 0$ and $\frac{\delta f}{\delta y} = 0$

$\frac{\partial f}{\partial x} = 0 \to$ that ,

$$\frac{1}{p}(x^p + y^p)^{\frac{1-p}{p}} p.x^{p-1} - \frac{1}{p}(1-x)^p + (1-y)^p)^{\frac{1-p}{p}} p.(1-x)^{p-1} = 0$$

$$(x^p + y^p)^{\frac{1-p}{p}} x^{p-1} = ((1-x)^p + (1-y)^p))^{\frac{1-p}{p}}(1-x)^{p-1} \Rightarrow$$

$$(x^p + y^p)^{\frac{1}{p}} (1-x) = ((1-x)^p + (1-y)^p))^{\frac{1}{p}} x$$

That is, $(x^p + y^p)^{\frac{1}{p}} - x (x^p + y^p)^{\frac{1}{p}} = x((1-x)^p + (1-y)^p))^{\frac{1}{p}}$

That is,

$$((1-x)^p + (1-y)^p))^{\frac{1}{p}} = (1-x)\left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}} = \left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}} - (x^p + y^p)^{\frac{1}{p}}, \text{ if } x \in (0, 1)$$

Hence , $f(x,y) = (x^p + y^p)^{\frac{1}{p}} + ((1-x)^p + (1-y)^p)^{\frac{1}{p}} = \left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}$ if $x \in (0, 1)$

Now,

$$f(1,1) = 2^{\frac{1}{p}} = f(0,0) \Rightarrow$$

$$= \left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}, \ x \in (0, 1), \Rightarrow$$

$$1 + (\frac{y}{x})^p \quad = 2 \Rightarrow \frac{y}{x} = 1. \Rightarrow \quad y = x.$$

■

The following generalizes Theorem 2.1

Theorem 2.4: Let $1 < p < \infty$, a,b $\in$ Rand f: $R^2 \to R$ be defined as,

f(x,y) = $(|x^p| + |y^p|)^{\frac{1}{p}} + (|(a-x)^p| + |(b-y)^p|)^{\frac{1}{p}}$, x between 0 and a and y between 0 and b . Then f is a constant if and only if x = $\frac{a}{b}$ y

Proof : First suppose that **x =** $\frac{a}{b}$y

Then,

$$f(x,y) = \left(\left|\frac{a}{b}\right|^p + 1\right)^{\frac{1}{p}} |y| + \left(\left|\frac{a}{b}\right|^p + 1\right)^{\frac{1}{p}} |b-y|$$

$$= \left(\left|\frac{a}{b}\right|^p + 1\right)^{\frac{1}{p}} |b|$$ (since, 0 < y < b or b< y < 0 , we have,

$$|y| + |b-y| = |b|$$

$$= (|a^p| + |b^p|)^{\frac{1}{p}} \quad \text{a constant.}$$

Conversely suppose that f is a constant.

Case 1: 0 < x < a, 0 < y < b

Then f(x,y) = $(x^p + y^p)^{\frac{1}{p}} + ((a-x)^p + (b-y)^p)^{\frac{1}{p}}$

Then $\frac{\partial f}{\partial x} = 0$ and $\frac{\delta f}{\delta y}$ =0

$\frac{\partial f}{\partial x}$ 0 imply that,

$$\frac{1}{p}(x^p + y^p)^{\frac{1-p}{p}} p.x^{p-1} - \frac{1}{p}(a-x)^p + (b-y)^p)^{\frac{1-p}{p}} p.(a-x)^{p-1} = 0$$

$$(x^p + y^p)^{\frac{1-p}{p}} x^{p-1} = ((a-x)^p + (b-y)^p))^{\frac{1-p}{p}} (a-x)^{p-1} \quad \Rightarrow$$

$(x^p + y^p)^{\frac{1}{p}}$ (a-x) = $((1-x)^p + (1-y)^p))^{\frac{1}{p}}\, x$

That is, $a\,(x^p + y^p)^{\frac{1}{p}}$ - x $(x^p + y^p)^{\frac{1}{p}}$ = $x\,((a-x)^p + (b-y)^p))^{\frac{1}{p}}$

That is,

$$((a-x)^p + (b-y)^p))^{\frac{1}{p}} = (a\text{-}x)\left(1 + (\frac{y}{x})^p\right)^{\frac{1}{p}} = a\left(1 + (\frac{y}{x})^p\right)^{\frac{1}{p}} - (x^p + y^p)^{\frac{1}{p}},$$

Hence , f(x,y) = $(x^p + y^p)^{\frac{1}{p}} + ((1-x)^p + (1-y)^p)^{\frac{1}{p}} = a\left(1 + (\frac{y}{x})^p\right)^{\frac{1}{p}}$

Now,

$$f(0,0) = (a^p + b^p)^{\frac{1}{p}} = (|a^p| + |b^p|)^{\frac{1}{p}} = f(a,b)$$

Hence,

$$(a^p + b^p)^{\frac{1}{p}} = a\left(1 + (\frac{y}{x})^p\right)^{\frac{1}{p}}$$

That is,

$$\frac{y}{x} = \frac{b}{a}$$

That is $\mathbf{x} = \frac{a}{b}y$

The other cases can be proved in similar way.

■

Corollary: Let $1 < p < \infty$, a, b, c, d $\in$ R and f: $R^2 \to R$ be defined as,

f(x,y) = $(|(a-x)^p| + |(b-y)^p|)^{\frac{1}{p}} + (|(x-c)^p| + |(y-d)^p|)^{\frac{1}{p}}$, x between a and c and y between b and d. Then f is a constant if and only if x - a = $\frac{c-a}{d-b}$ y.

The above results prove that on R^2 induced by the norm $\|.\|_p$,$1 < p < \infty$, for x, y in R^2 , Seg (x,y) = $\{ x + (1-t)(y-x) : t\in (0,1)\}$. Hence we have proved the following Theorem.

■

Theorem 2.5: Let C be the interval convexity on R^2 induced by the norm , $\|.\|_p$ $1< p < \infty$.

Then C coincides with the Euclidean convexity.

Theorem 2.6. Let C be the interval convexity on R^2 induced by the norm , $\|.\|_\infty$

Then C does not coincides with the Euclidean convexity.

Proof : Let x = (0, 0), y = (1, 0) Then,

$\|x-y\|_\infty = 1$

Let z = $(\frac{1}{2},\frac{1}{2})$. Then z is not on the line joining x and y. But

$\|x-z\|_\infty + \|z-y\|_\infty = \frac{1}{2}+\frac{1}{2} = 1$ Hence z belong to the convex hull of {x,y}. Thus the line segment joining x and y is not a member of C..

■

Theorem 2.7. : Let C be the interval convexity on R^2 induced by the norm , $\|.\|_\infty$. Let x = (0, 0) and y = (a, b). and let $\alpha = \max\{|a|, |b|\}$ Then, The convex hull of { x, y} is the area bounded by the straight lines y = ±x, y = x + b – a and y+x = b-a

Proof:

Case1 **:** (a, b) is in the first quadrant. Then a, b ≥ 0.

Let a ≥ b.

Then $\|x-y\|_\infty = a$

A point z = (s, t) is in the convex hull of x and y if and only if

Max $\{ |s|, |t| \}$ + Max $\{ |a-s|, |b-t| \}$ = a ,that is if and only if

$|t| \leq |s|$, and $|b\text{-}t| \leq |a\text{-}s|$ that is if and only if $-s \leq t \leq s$ and $s-a \leq b-t \leq a-s$, that is if and only if $-s \leq t \leq s$ and $b-a+s \leq t \leq a+b-s$.

Now let $b \geq a$. Then, $\|x-y\|_\infty = b$ and a point z = (s, t) is in the convex hull of x and y if and only if

Max { $|s|$, $|t|$ } + Max { $|a\text{-}s|$, $|b\text{-}t|$ } = b ,that is if and only if

$|s| \leq |t|$, and $|a\text{-}s| \leq |b\text{-}t|$ that is if and only if $-t \leq s \leq t$ and $t-b \leq a-s \leq b-t$'

That is $-t \leq s \leq t$ and $a-b+t \leq s \leq a+b-t$

Case2 **:** (a, b) is in the II quadrant. Then $a \leq 0$, $b \geq 0$. Then,

Let $|a| \geq b$.

Then $\|x-y\|_\infty = -a$

A point z = (s, t) is in the convex hull of x and y if and only if

Max { $|s|$, $|t|$ } + Max { $|a\text{-}s|$, $|b\text{-}t|$ } = Max { -s, $|t|$ } + Max {s-a , $|b\text{-}t|$ }= - a, that is if and only if

$|t| \leq -s$, and $|b\text{-}t| \leq |a\text{-}s|$ that is if and only if $s \leq t \leq -s$ and $a-s \leq b-t \leq s-a$, that is if and only if $-s \leq t \leq s$ and $a+b-s \leq t \leq b-a+s$.

Similarly, the other cases can be proved.

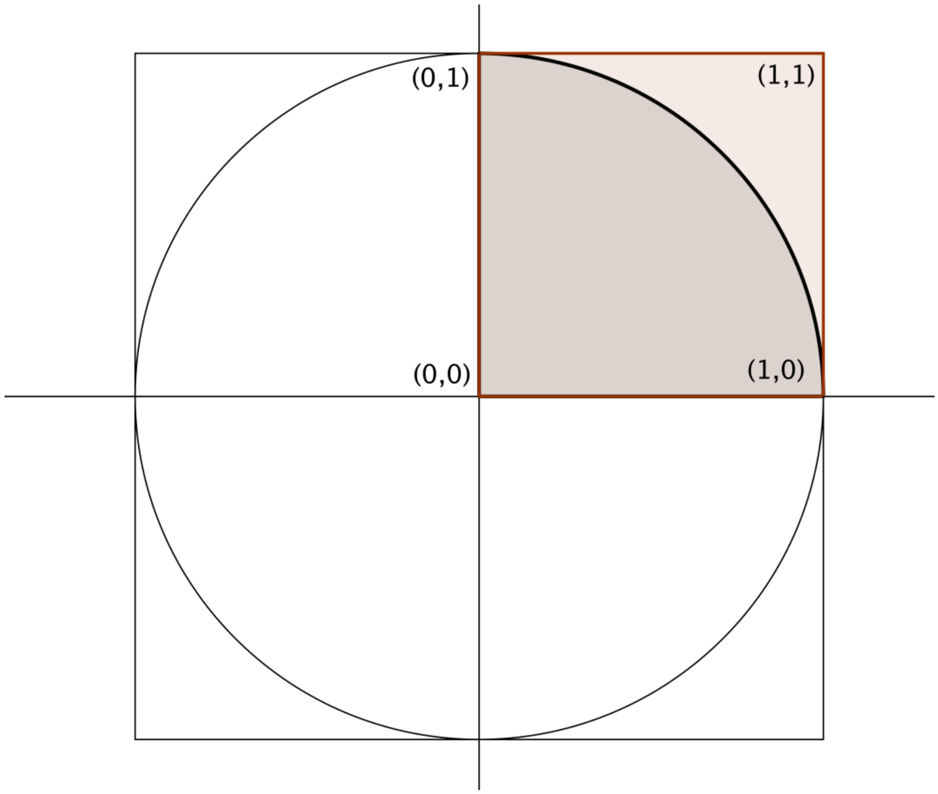
■

Corollary: The convex hull of any two points (a, b) and (c,d) is the rectangle bounded by the straight lines $|a\text{-}x| = |b\text{-}y|$ and $|c\text{-}x| = |d\text{-}y|$

1. ***Theorem2.8:*** The area bounded by a circle C in R^2 is convex with respect the convexities induced by $\|.\|_p$, $1 < p < \infty$. But not convex with respect to the convexity induced by $\|.\|_\infty$ or $\|.\|_1$

Proof : With respect to convexities induced by $\|.\|_p$, $1 < p < \infty$., it is clear. Now let R^2 be given the norm $\|.\|_1$. Consider the set, S = { (x, y): $x^2 + y^2 \leq 1$ }.

Then $z_1 = (1, 0)$, and $z_2 = (0, 1)$ are in S. Also we can see that $(1, 1) \in Co(\{z_1, z_2\})$, but (1, 1)is not appoint in S. Hence S is not convex with respect to $\|.\|_1$.



Similarly if R^2 is given the norm $\|.\|_\infty$, and Let $S' = \{ (x, y): \ x^2 + y^2 \leq 2 \}$.

Then take $z_1 = (1, -1)$ and $z_2 = (1, 1)$. Then z_1 and z_2 are in S' ,

(2, 0) is in the convex hull of z_1 and z_2 , but (2, 0) is not in S.

■

Note : If the points A and B are on the line y = x or y = -x , Then the convex hull is the line segment joining them.

Hahn –Banach Seperation Theorem **[1]** : Let E_1, E_2 be nonempty disjoint convex subsets of a normed space X over a field K with E_1 open. Then there is a Real hyperplane in X which separates E_1 and E_2 in the sense that: there is some $f \in X'$ and $t \in R$ such that

$$\text{Re } f(x_1) < t \leq \text{Re } f(x_2)$$

for all $x_1 \in E_1$ and $x_2 \in E_2$

■

Then the statement in the following Remark is true.

Remark: Let E be a nonempty convex subset of a normed space X over K. If $a \in X$, $a \notin \bar{E}$ then there are $f \in X'$ and $t \in R$ such that

$$\text{Re } f(x) \leq t < \text{Re } f(a) \text{ , for all } x \in E.$$

If $E^o \neq \phi$ and b is in the boundary of E in X, then there is a nonzero $f \in X'$ such that

$\text{Re } f(x) \leq \text{Re } f(b)$, for all $x \in \bar{E}$.

Thus we can see that any (Euclidean) convex set is the intersection of half spaces. Hence it satisfies all the separation axioms.

Remark: When consider the interval convexity all hyperspace need not induce half spaces.

For example, Let $\mathcal{C}$ be the interval convexity on R^2 induced by the norm , $\|.\|_\infty$.Then straight lines are not in general convex. The only half spaces are those determined by the hyperspaces $y = \pm x$. Similarly when $\|.\|_1$ is considered, the straight lines y = 0 and x = 0 induce half spaces.

Theorem 2.9) : In an inner product space X, the convexity induced by the norm coincide with the usual convexity.

Proof: Let $\mathcal{C}$ be the convexity induced by the norm and $\mathcal{D}$ denote the usual convexity.

Let $C \in \mathcal{C}$., $x, y \in C$ $t \in (0, 1)$ and let $z = tx + (1-t)y$.

$\|x - z\| = \|(1-t)x - (1-t)y\| = (1-t)\|x - y\|$. Similarly

$\|z - y\| = \|t\,x - ty\| = t\,\|x - y\|$.

That is, $\|x - y\| = \|x - z\| + \|z - y\|$

Hence, z ∈ C. That is C ∈ D .

Conversely suppose that C ∈ D, x, y ∈ C and z be such that $\|x - y\| = \|x - z\| + \|z - y\|$

Then by Theorem 2.2,

Then, $x - z = k(z - y)$ for some $k \geq 0$

That is, $z = \frac{1}{1+k}x + \frac{k}{1+k}y$. Put $= \frac{1}{1+k}$. Then $t \in (0,1)$ and z = tx + (1-t)y . and hence z ∈ C.

That is C ∈ C..

2.3. Convexity parameters

The convexity Parameters such as the the Helly number, Caratheodory number and Radon numbers are studied for interval convexities and found that

1. Helly number of the convexity induced by $\|.\|_\infty$ or $\|.\|_1$ on R^2 is 3 as that with usual convexity .
2. The carathedory number is 3.
3. The radon number is also 3

It can be seen that , If **h, c, r** denote the Helly number, Caratheodory number, Radon number of X with usual convexity and $\mathbf{h'}$ denote that with interval convexity. It is Clear that

i. $\mathbf{h' \leq h}$

ii. $\mathbf{r' \leq r}$

But regarding the Carathedory number, an easy conclusion is not possible, The structure of intervals are different . However it is an easy observation that this convexity is join hull commutative.

H- CONVEXITY.

3.1. H- convexity on R^n

The separation axioms and their impact on the developments of theory convexity led to the study of half spaces. We have already noted that for any linear functional f on a Real linear space, and $\alpha \in R$,

$$\{ x \in X : f(x) > \alpha \}, \{ x \in X : f(x) \geq \alpha \}, \{ x \in X : f(x) < \alpha \} \text{ and } \{ x \in X : f(x) \leq \alpha \}$$

are half spaces. Thus it is observed that a collection $\mathcal{F}$ of linear functional give rise to a collection of half spaces and it generates a convexity on X precisely the smallest convexity containing all the half spaces corresponding to the linear functional in $\mathcal{F}$. This is termed as the H- convexity symmetrically generated by $\mathcal{F}$.

Definition: Let X be a linear space over R and $\mathcal{F}$ be a family of linear functional on X. The convexity $\mathcal{C}$ on X generated by $\{ f^{-1}(a, \infty): f \in \mathcal{F}$ and $a \in R \}$ is called the H convexity generated by $\mathcal{F}$. If $-f \in \mathcal{F}$ whenever $f \in \mathcal{F}$, then $\mathcal{C}$ is called a symmetric convexity. We usually omit one of f, -f and say that $\mathcal{F}$ symmetrically generates $\mathcal{C}$.

***Example1*:** Let X be an inner product space and C be a convex set in X. By Hahn -Banach separation Theorem, Any convex set can be separated from any point not in C by hyper plane. That is, for any $x \in X$, $x \notin C$, there is a half space H such that $C \subset H$, but $x \notin H$, Thus C is the intersection of half spaces. Thus the convexity on X is the H-Convexity generated by the set of all bounded linear functionals.

Example 2: Let $X = R^2$. Let f_1, f_2, f_3 and f_4 be defined as follows

$f_1(x, y) = x – y$, $f_2(x, y) = x + y$, $f_3(x, y) = x$, $f_4(x, y) = y$. Then $\{ f_1, f_2, -f_1, -f_2 \}$ generates the convexity induced by the norm $\|.\|_\infty$, and $\{ f_3, f_4, -f_3, -f_4\}$ generates the convexity induced by the norm $\|.\|_1$.

Example 3: Let $X = R^n$, $n \geq 3$ with the norm induced by the inner product defined by

$$\langle x, y\rangle = x(1)y(1) + x(2)y(2) + \ldots\ldots\ldots\ldots+x(n)y(n)$$

For $t = (t_1, t_2,\ldots..t_{n-1}) \in R^{n-1}$, define f_t on R^n by,

$$f_t(x) = t_{1\ x(1)\ +}\ t_2x(2)+\ldots\ldots..t_{n-1}x(n-1) – x(n)$$

Then f_t is linear for all $t \in R^{n-1}$.

Let $r > 0$ and $T = \{ t = (t_1, t_2,\ldots..t_{n-1}) \in R^{n-1} : \|t\| = r\}$, and $S = \{ (t, 0) : t\in T\}$

Then for any $x \in S$, $|f_t(x)| \leq \|t\| \|x\| \leq r^2$.

Now for, $x \in S$, $f_t(x) = r^2$ if and only if $x = (t, 0)$, for, by Schwartz's inequality,

$|f_t(x)| = \|t\| \|x\|$ if and only if $\langle x, (t, 0)\rangle = \|x\| \|(t, 0)\|$ if and only if x and (t, 0) are linearly dependent. Since $\|x\| = \|(t, 0)\| = r$, $x = \pm (t, 0)$.

Now If $x = - (t, 0)$, Then $f_t(x) = -r^2$. Hence, $x = (t, 0)$.

Hence, The hyper plane $f_t(x) = r^2$ meets S in only one point (t, 0). Similarly the hyper plane

$f_t(x) = - r^2$ meets S in only one point (-t, 0). Hence,

S is contained in the half space $\{ x \in R^n : f_t(x) \leq r^2\}$ and also contained in $\{ x \in R^n : f_t(x) \geq - r^2\}$

Let $C = \bigcap_{t\in T} \{ x \in R^n : -r^2 \leq f_t(x) \leq r^2\}$.

Then C is convex and $S \subset C$. Let $\mathcal{C}$ be the H convexity on R^n symmetrically generated by

$\mathcal{F} = \{ f_t : t \in T\}$.Then C is the convex hull of S.

Now consider the points $c = (0,0,\ldots,r^2)$ and $y' = (0,0,\ldots,-r^2)$ .Then c and c' are in C. Now Let

$C' = C \sim \{ c, c'\}$. Then $C' \notin \mathcal{C}$.

For $t \in T$, Let P_t denote the hyper plane $f_t(x) = r^2$ and P_{-t} denote the hyper plane $f_t(x) = -r^2$

Now, let E be any finite subset of C'. Then there are $s,t \in T$ such that $E \cap P_t = \phi$ and $E \cap P_{-s} = \phi$

Thus, $E \subset \{ x \in R^n : f_t(x) < r^2\} \cap \{ x \in R^n : f_t(x) > - r^2\}$.

Hence $c, c' \notin co (E)$.

Hence $co (E)$ is $\subset C'$ for any finite subset of C'. But C' is not convex, for,

$S \subset C'$ and $c,c' \in co (S)$.

Thus H is of infinite arity.

Remark: $\mathcal{C}$ is not join hull commutative . See the case when n = 3, r = 1

Then $\mathcal{F} = \{ f_t : ft(x, y, z) = t_1 x+ t_2 y - z ; \; t_1{}^2 + t_2{}^2 = 1\}$

Let $ax + by = 0$, $z = 0$ $(a, b) \neq (0, 0)$ be a straight line.

Let $t_1 = \frac{a}{\sqrt{a^2+b^2}}$ and $t_2 = \frac{b}{\sqrt{a^2+b^2}}$

Then the hyper planes,

$t_1 x+ t_2 y - z = 0$ and $-t_1 x+ -t_2 y - z = 0$ intersect at the line $ax + by = 0$, $z = 0$. Hence every line in the x-y plane is convex (w.r. to the H- convexity generated by $\mathcal{F}$ }.

Similarly we can see that any line segment in the x-y Plane is convex.

Let C be the segment joining any two distinct points A and B in the x-y plane and p be a point not in the line joining A and B .Then,

$F = \cup \{ Co (\{c, p\}) : c \in C\}$ is the area in the plane bounded by the triangle with vertices A, B and p. But F is not convex. The following figure gives the convex hull of the triangle.

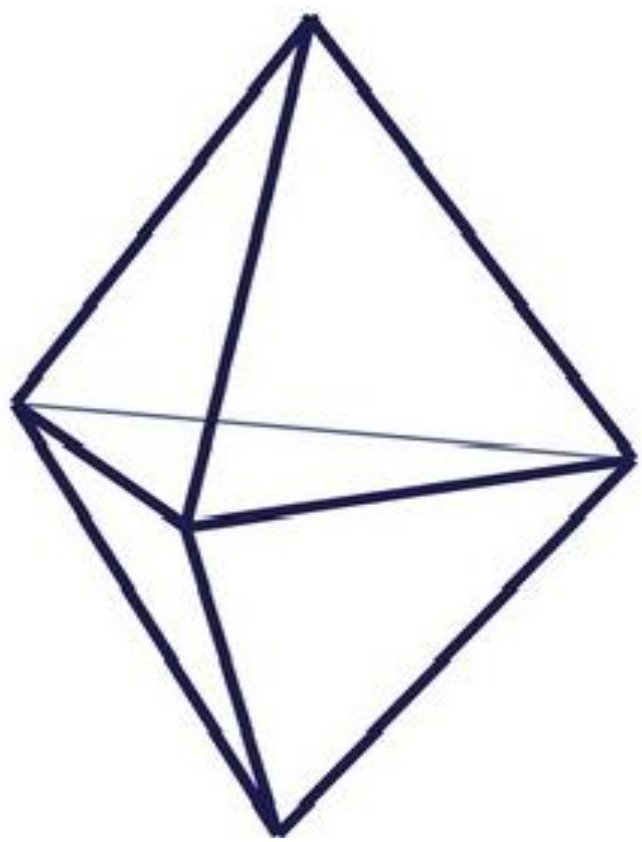


Figure 4

3.2. Separation Properties

Since the H convexity is generated by half spaces, it is clearly S_3. But it is not S_4 in general. For example, Let C be the H-convexity symmetrically generated by the co- ordinate projections and their sum, defined on R^3. $F = \{ f_1 , f_2, f_3, f_4 = f_1 + f_2 + f_3 \}$

Let $C_1 = \{ (x, y, z) : x \leq 0 \text{ and } y \leq 0 \}$ and

$C_2 = \{ (x, y, z) : z \leq -1 \text{ and } x + y + z \geq 0 \}$

Then C_1 and C_2 are disjoint convex sets which can not be separated by half spaces. That is the convexity is not S_4. However we can see that,

Remark: 1) Any symmetric H- Convexity on R^2 is S_4.

2) The Symmetric H- convexity defined by Example 3 is not S_4. For if we take n = 3

Let $C_1 = \{ (x, y, z) : z = 0 \text{ and } y = 0 \}$ and

$C_2 = \{ (x, y, z) : z = 1 \text{ and } x = 0 \}$. Then C_1 and C_2 are disjoint convex sets which can not be separated by half spaces. That is the convexity is not S_4.

3.3. Convexity Parameters

Since any H convex set in R^n is Euclidean convex , it can be seen that,

$h \leq n+1$ and

$r \leq n+1$

But for the H- convexity in Example 3, The caratheodory number is ∞.

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CONCLUDING REMARKS AND SCOPE FOR FURTHER STUDY

This work is an attempt to find out the convexity properties of Normed spaces. In this we analyze different convex structures on normed spaces. In a normed space there is the convexity associated with the linear space structure and also convexity induced by the interval function defined by the norm. It is found that there is difference between the two. Also The H convexity symmetrically generated by a family of linear functional is also studied. It is also observed that in linear spaces of finite dimension greater than 2, there are convexity of infinite arity. An example is given to show that the H- Convexity need not be join hull commutative. Also the study of the parameters such as Helly number, Caratheodory number, Radon number and has been done. Some of the findings are given below

2. Let $\mathcal{C}$ be the interval convexity on R^2 induced by the norm , $\|.\|_\infty$. Let x = (0, 0) and y = (a, b). and let α = max{ | a | , | b | } Then, The convex hull of { x, y} is the area bounded by the straight lines y = ±x, y = x + b – a and y+x = b-a
3. Let $\mathcal{C}$ be the interval convexity on R^2 induced by the norm , $\|.\|_p$ $1< p < \infty$.

   Then $\mathcal{C}$ coincides with the Euclidean convexity.
4. The area bounded by a circle C is convex with respect to both the convexities
5. If x = (x(1), x(2)) and y = (y(1), y(2)),

   Co (x, y) = {z: z(1) is between x(1) and y(1), z(2) is between x(2) and y(2)}
6. Let $X = R^n$, $n \geq 3$ with the norm induced by the inner product defined by

   $\langle x, y\rangle$ = x(1)y(1) + x(2)y(2) + ………….+x(n)y(n)

   For , $t = (t_1, t_2,\ldots..t_{n-1}) \in R^{n-1}$, define f_t on R^n by,

   $f_t(x) = t_{1\,x(1)} + t_2x(2)+\ldots\ldots..t_{n-1}x(n-1) - x(n)$ and

   Then The H-convexity defined by

$\mathcal{F}$ = { f_t : ft(x$_{(1)}$, x(2) ……x(n)) = $t_{1\,x(1)} + t_2x(2)+\ldots\ldots..t_{n-1}x(n-1) - x(n)$; $t_1{}^2 + ..t_{n-1}{}^2 = 1$} is of infinite arity.

This is far away from being complete, because in several cases we could consider the dimension to be 2. We mention some of the problems that are to be considered in future.

1. Distinguish and compare the convexities induced by different norms on infinite dimensional Normed spaces.
2. Characterize the symmetric H-convexity which is S_4.
3. Study other convexity related concepts for higher dimensional Normed spaces.

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