D 13189 (Pages : 3) Name

Reg. No.................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2016

(CUCSS)

Mathematics

MT 1C 03—REAL ANALYSIS—I

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer all *questions.*
Each question has weightage 1.

1. Let X be a metric space. Define neighbourhood of $p \in X$ and prove that it is an open set.

2. Construct a bounded set of real numbers with exactly two limit points.

3. Let $E = \{p \in Q : 2 < p^2 < 3)$. Show that E is closed and bounded in Q. Is E compact.

4. Let $f : [0, 2\pi) \to \{(\cos t, \sin t) : t \in [0, 2\pi)\}$ be defined by $f(t) = (\cos t, \sin t)$. Verify whether f is a homeomorphism.

5. Identify the type of discontinuity at rational points $x = \frac{m}{n}$ (where m and n are integers without any common divisors) of the function f defined by $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

6. Let f be defined on [0, 2] by $f(x) = \frac{x}{x+1}$. If f uniformly continuous. Justify your answer.

7. If f is a real differentiable function defined on $[a, b]$ and $f'(a) < c < f'(b)$, prove that there is a point $x \in (a, b)$ such that $f'(x) = c$.

8. Prove that $\lim_{x \to 0} \sin x = 0$.

Turn over

9. Explain whether the Mean Value theorem is applicable to $f(x) = 2 + (x-1)^{\frac{2}{3}}$ in [0, 2].

10. If f is bounded on [a, b], f is continuous at s where a $< s < b$ and a (x) = I $(x - s)$ where I is the unit step function defined by $\mathbf{I}(x) = 0$ if x 0 and $\mathbf{I}(x) = 1$ if x > 0, then prove that $\int_a^b f d\alpha$ $f(s)$.

11. If f is continuous on $[a, b]$ and a has a continuous derivative on $[a, b]$ then prove that $\int_a^b f d\alpha = \int_a f \alpha' dx$.

12. Define a rectifiable curve. Give an example of a rectifiable curve.

13. Let $\{f_n\}$ be a sequence of functions differentiable on [a, b] such that $\{f_n(x_0)\}$ converges for some point x_0 on [a, b]. If $\{f'_n\}$ converges uniformly on $[a, b]$, prove that $\{f_n\}$ converges uniformly on $[a, b]$.

14. Define equicontinuous family of functions. State a sufficient condition for a family $\{f_n\}$ of continuous functions defined on a compact metric space K to be equicontinuous on K.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions from the following ten questions.*
Each question has weightage 2.

15. Prove that the interior of a set S is the largest open subset of S.

16. Define a compact set. Explain with details why S = (0, 1) is not compact.

17. If F is closed and K is compact prove that F n **K** is compact.

18. If $f : X \to Y$ is continuous where X is a compact metric space and Y is a metric space, prove that f is uniformly continuous on X.

19. Prove that monotonic functions have no discontinuities of the second kind.

20. State Taylors theorem. Illustrate with an example.

21. If f is bounded in [a, b], f has only finitely many points of discontinuities on [a, b] and a is continuous at every point at which f is discontinuous, prove that $f \in \mathcal{R}(\alpha)$ on [a, b].

22. If f is Riemann integrable on $[a, b]$ and if there exists a differentiable function F such that $F' = f$, prove that $\int_a^b f(x)\,dx = F(b) - F(a)$.

23. If $\{f_n\}$ is a sequence of continuous functions defined on E in a metric space and if f_n f uniformly on E prove that f is continuous on E.

24. If f_n f uniformly on $[a, b]$, f_n E R (a) on [a, b] where a is monotonically increasing on $[a, b]$, prove that $f\, E\, \mathcal{R}$ (a) on [a, b] and $\int_a^b f d\alpha = \lim_{n \in w} \int_a^b f_n d\alpha$.

(7 x 2 = 14 weightage)

Part C

Answer any two questions from the following four questions.
Each question has weightage **4.**

25. (a) Let X be a compact set in R and A c X be closed. Prove that A is compact.

(b) Let X and Y be metric spaces. Prove that $f : X \to Y$ is continuous if and only if f^{-1} (C) is closed in X for every closed set C in Y.

26. Let a increases monotonically and a' E on $[a, b]$. If f is a bounded real function on $[a, b]$ prove that $f \in \mathcal{R}$ (a) on $[a, b]$ if and only if fa' E $\mathcal{R}$. Also prove that $\int_a^b f d\alpha \quad \int_a^b f(x)$ a' (x) dx.

27. Prove that there exists a real continuous function on the real line which is nowhere differentiable.

28. State and prove the Stone-Weierstrass theorem.

(2 x 4 = 8 weightage)