D 6716 **(Pages : 3)** **Name...**

Reg. No

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2016

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS

Time : Three Hours **Maximum : 36 Weightage**

Part A

Short answer questions **(1-14).**
Answer **all** *questions.*
Each question has 1 weightage.

1. **For 0** $< p < 1$, **define II** $\|p :$ $\rightarrow$ **R by** :

$$x \quad = \left(\sum_{j=1}^{n} \left| x(j)^{-} \right| \right)^{-}.$$

Is II x $\|_p$ **a norm on** $\mathbb{R}^n$ **? Justify your answer.**

2. **Prove that the closure of a subspace of a normed space is a normed space.**

3. **Let X be a normed space. If** E_1 **is open in X and** E_2 **C X, then prove that** $E_1 + E_2$ **is open in X.**

4. **Let E be a convex subset of a normed space X. Prove that the closure** E **of E is a convex set.**

5. **Let X be a linear space of all polynomials in one variable with coefficients in C . For** p e **X with**

$p(t) = \quad + a_1 t + \ldots + a_n t$, **let :**

p II **sup** P (t) I : 0 t _1) **and** $P \quad = | a_0 | + | a_1 | + \ldots + | a_n |$.

Prove that II p ,

6. **Prove that there exists a discontinuous linear map from** $/^2$ **into itself.**

7. **Let** $\langle \; , \rangle$ **be an inner product in a linear space X and let** x E X . **Prove that** *(x, y)* = **0 for all y** E X ,
if and only if x = **0.**

8. **Let X be an inner product space with inner product** $\langle \quad \rangle$. **II** $x_n - x$ $\rightarrow$ **0 and II** y_n **Y** $\|$ › **0 then**
prove that $(x_n, y_n) \rightarrow (x, y)$.

Turn over

9. Let E be an orthonormal set in an inner product space X. Prove that II x ——=—2 ~~for all~~ x, y E E with $x \neq y$.

10. Let $u_n(t) - \frac{\sin nt}{\sqrt{\pi}}$ where t E $[-\pi, \pi]$. **Prove that** $\{u_1$, **u2, ...I** is an orthonormal set in $L^2([--a, \pi])$.

11. State Hahn-Banach Separation Theorem.

12. Give an example of a normed space which is not a Banach space.

13. Prove that in a Banach space X every absolutely summable series of elements in X is summable in X.

14. Define Schauder basis and give an example.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *from the following ten questions* (15-24).
Each question has weightage 2.

15. Let X be a separable metric space and let $Y \subset X$. Prove that Y is separable.

16. Let x, y be measurable functions on a measurable subset E of R, let $0 < p < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Prove that :

$$\int_E |xy| \, d \leq \left(\int_E x \text{ IP} \, dm \right)^{\frac{1}{p}} (\text{S}_E \, I \, y \quad d \qquad 1$$

17. Prove that finite dimensional subspaces of a normed space are closed.

18. Prove that a linear map F from a normed space X onto a normed space Y *is* a homeomorphism if there are a, $\beta > 0$ such that :

a x F *(x)* ‖ 5- $\beta \| x \|$ for all x E X .

19. Let $\langle \, , \rangle$ be an inner product on a linear space X. For all x, y E X, prove that :

$$4(x, y) = (x + y, \ x + y) — (x - y, x - y) + \ (x + iy, x + iy\rangle - i \, (x - iy, x - iy\rangle.$$

20. Let $\{u_\alpha\}$ be an orthonormal basis in a Hilbert space H. For x E H, prove that :

$$x = \sum (x, \ u_n) \, u_n \text{ where } \{u_1, u_2, \quad = \{u_\alpha : (x, u_\alpha) \neq \quad .$$

21. Let E and F be closed subspaces of a Hilbert space H and E 1 F. Prove that E + F is a closed subspace of H.

22. Let E be a non-empty convex subset of a formed space X over a field K. If $E^{\circ} \neq 0$ and b belong to the boundary of E in X, then prove that there is a non-zero bounded linear functional f on X such that Re $(x) \leq \text{Re} f(b)$ for all x E .

23. Let X be a normed space and let Y be a dense subspace of X. If g is a continuous linear functional on Y $(g \in Y')$, then prove that there is a continuous linear functional f on X such that $f_Y = g$.

24. Let Y be a closed subspace of a Banach space X. Prove that X/Y is a Banach space.

(7 x 2 = 14 weightage)

Part C

Answer any two from the following four questions (25-28).
Each question has weightage 4.

25. For $1 < p < co$, prove that the metric space $1P$ is complete.

26. Show that a non-zero Hilbert space H is separable if and only if H has a countable orthonormal basis.

27. Prove that every normed space can be embedded as a dense subspace of a Banach space.

28. State and prove Uniform Boundedness principle.

(2 x 4 = 8 weightage)