CHAPTER I

I

INTRODUCTION

The concept of *convexity* which was mainly defined and studied in R^n in the pioneering works of Newton, Minkowski and others as described in [18], now finds a place in several other mathematical structures such as vector spaces, posets, lattices, metric spaces and graphs. This development is motivated by not only the need for an abstract theory of convexity generalising the classical theorems in R^n due to Helly, Caratheodory etc., but also to unify geometric aspects of all these mathematical structures. In the course of the development it is found that the properties of convex sets have been analyzed mainly in three ways, qualitatively, quantitatively and combinatorially and finds its applications in problems of pattern recognition, optimization, etc. [68].

The theory of *graphs* which originated in the solution of the famous Königsberg bridge problem during 1736 by Leonard Euler, now finds quite a lot of applications in many other branches of science, engineering and social science. See [5], [6], [10] for details.

This thesis is an attempt to study mainly some combinatorial problems of convexity spaces and graphs, following the footsteps of Levi, Jamison, Sierksma, Soltan, Duchet and others.

1.1 DEFINITIONS AND PRELIMINARIES

In this section, we consider some basic definitions and concepts mainly from [2], [7], [8] and [12]. For notations and terms not mentioned here, we follow [7], [8] and [12].

By a graph $G = G(V,E) = G(p,q)$ we generally mean a finite connected graph without loops and multiple edges, with vertex set V, edge set E, of order p and size q. The symbol $\langle S \rangle$ means the subgraph induced by S.

Definition 1.1. Let $G = (V,E)$ be a graph. $d(u,v)$, the distance between u and v in $V(G)$ is the length of the shortest path connecting u and v, the eccentricity of the

vertex u, $e(u) = \max\{d(u,v): v \in V(G)\}$,
$diam(G) = \max\{e(u): u \in V(G)\}$, $rad(G) = \min\{e(u): u \in V(G)\}$, $C(G) = \{u: e(u) = rad(G)\}$ the center of G and a graph G is called *self centered* if $C(G) = V(G)$.

Definition 1.2. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G_1 \times G_2$ of G_1 and G_2 is defined as the graph G where $V(G) = V_1 \times V_2$ and (u_1, v_1) is adjacent to (u_2, v_2) if either $u_1 = u_2$ and $v_1 v_2 \in E_2$ or $u_1 u_2 \in E_1$ and $v_1 = v_2$. The *join* $G_1 + G_2$ is obtained by joining all the vertices of G_1 to all the vertices of G_2. The *sequential join* $G_1 + G_2 + \ldots + G_n$ of $G_1, G_2, \ldots, G_n$ is obtained by joining all vertices of G_i to all vertices of G_{i+1} for $i = 1, 2, \ldots, n-1$. The graph $S_{m,n} \simeq \bar{K}_m + K_1 + K_1 + \bar{K}_n$ is called a double star.

Definition 1.3. A chord of a cycle C is an edge connecting non consecutive vertices of C. A graph G is *chordal* if every cycle of length at least four has a chord.

Definition 1.4. A graph G is *Ptolemaic* if for any $u, v, w, x \in V(G)$,

$$d(u,v).d(w,x) \leq d(u,w).d(v,x) + d(u,x).d(v,w).$$

Definition 1.5. The size of the maximum clique in G is the *clique number* $\omega(G)$ of G. $S \subset V(G)$ is said to separate u,v in V(G) if u and v lie in different components of $G \setminus S$. S is a *clique separator* whenever S induces a clique in G.

Definition 1.6. Let X be a set. Then $I: X \times X \rightarrow \mathcal{P}(X)$ is an *interval* function on X if the following conditions hold.

(a) $a,b \in I(a,b)$ - Extensive law.

(b) $I(a,b) = I(b,a)$ - Symmetry law.

Definition 1.7. Let $G = (V,E)$ be a graph. $S \subseteq V$ is *geodesically convex* if for all x,y of S, $I(x,y) = \{z : z$ is on some shortest x-y path$\} \subseteq S$. These convex sets are also called distance convex (d-convex) sets. $S \subseteq V$ is *minimal path convex* (m-convex) if for all x,y of S, $I(x,y) = \{z : z$ is on some chordless x-y path$\} \subseteq S$.

Definition 1.8. For a graph G, V(G), ϕ and $S \subseteq V(G)$ whose induced subgraphs are isomorphic to K_n for $n > 0$ are called trivial convex sets. For any integer $k \geq 0$ a graph G is k-convex if it has exactly k nontrivial convex sets. A (k,ω)-convex graph is a k-convex graph with clique number ω. (0,2)-convex graphs are called *distance convex simple (d.c.s)* if the convexity is geodesic convexity and *m-convex simple (m.c.s.)* if the convexity is m-convexity. When k=1 the k-convex graphs are called *uniconvex* graphs.

Definition 1.9. A graph is *convex simple* if it is either d.c.s or m.c.s.

Definition 1.10. A graph G is *interval monotone* if I(u,v) is convex for each pair of vertices u and v of G. It is *totally non interval monotone* (t.n.i.m.) if no nontrivial interval is convex. Here, the trivial intervals are those I(a,b) for which a=b, a adjacent to b or I(a,b) = V(G).

Definition 1.11. Let G = (V,E) be a graph and $S \subseteq V(G)$. Then the closure of S, (S) = {x:x is on some shortest path connecting vertices of S}. Then, define S^k as follows. $S^1 = (S)$, $S^k = (S^{k-1})$. If $S^k = S^{k+1}$ then S^k is convex. The geodetic iteration number gin(S) is the smallest number n such that $S^n = S^{n+1}$. The *geodetic iteration number* gin(G) is defined as the maximum value of a gin S over all $S \subseteq V(G)$.

Definition 1.12. A family $\mathscr{C}$ of subsets of a nonempty set X is called a *convexity* on x if

1) ϕ, $X \in \mathscr{C}$
2) $\mathscr{C}$ is stable for intersection, and
3) $\mathscr{C}$ is stable for nested union.

$(X, \mathscr{C})$ is called a *convexity space* and members of $\mathscr{C}$ are called convex sets. The smallest convex set containing a set A is called *convex hull* of A, denoted by Co(A).

Definition 1.13. A convexity space X is an *interval convexity space* if its convexity is induced by an interval.

Definition 1.14. A convexity space is of *arity* $\leq n$ if its convex sets are determined by n-polytopes. That is, a set C is convex if and only if $Co(F) \subseteq C$ for each subset F of cardinality at most n.

Definition 1.15. A convexity space X is a *matroid* if it satisfies the exchange axiom $A \subseteq X$ and $p, q \in X \setminus Co(A)$, then $p \in Co(\{q\} \cup A)$ implies that $q \in Co(\{p\} \cup A)$ and is an *antimatroid* (convex geometry) if it satisfies the antiexchange law, $A \subseteq X$, $p, q \in X \setminus Co(A)$ then, $p \in Co(\{q\} \cup A)$ implies that $q \notin Co(\{p\} \cup A)$.

Definition 1.16. A subset H of X is called a half space if both H and $X \setminus H$ are convex. A convexity space X is said to have *separation property*

S_1: if all singletons are convex.

S_2: if any two distinct points are separated by half spaces. That is, if $x_1 \neq x_2 \in X$ then there is a half space H of X such that $x_1 \in H$ and $x_2 \notin H$.

S_3: if any convex set and any singleton not contained in C can be separated by half spaces. That is, if $C \subseteq X$ is convex and if $x \in X \setminus C$, then there is a half space H of X such that $C \subseteq H$ and $x \notin H$.

S_4: if any two disjoint convex sets can be separated by half spaces. That is if $C_1, C_2 \subseteq X$ are disjoint convex sets then there is a half space H of X such that $C_1 \subseteq H$ and $C_2 \subset X \setminus H$.

Definition 1.17. A subset S of an interval space X is *star shaped* at a point $p \in S$ provided for every $x \in S$,

$I(x,p) \subseteq S$. The *star center* of S is the set of all points at which S is star shaped. X is said to have the *Brunn's property* if the star center of each subset of X is convex. The star center is also called the kernel of S, denoted by Ker(S).

Definition 1.18. Let X be convexity space then,

1. The *Helly number* of X is the smallest 'n' such that for each finite set $F \subset X$ with cardinality at least n+1, $\cap \{Co(F \setminus \{a\}) : a \in F\} \neq \phi$
   (that is, F is Helly (H-) dependent).

2. The *Caratheodory number* of X is the smallest number 'n' such that for each $F \subset X$ with cardinality at least n+1, $Co(F) \subset \cup \{co(F \setminus \{a\}) : a \in F\}$
   (that is, F is Caratheodory (C-) dependent).

3. The *Radon number* of X is the smallest number 'n' such that each $F \subset X$ with cardinality at least n+1, can be partitioned into two sets F_1 and F_2 such that
   $Co(F_1) \cap Co(F_2) \neq \phi$
   (that is, F is Radon (R-) dependent).

4. The exchange number of X is the smallest number n such that for each $F \subset X$ of cardinality at least n+1 and for each $p \in F$, $Co(F \setminus \{p\}) \subset \cup \{Co(F \setminus \{a\}) : a \in F, a \neq p\}$ (that is, F is exchange (E-) dependent).

These numbers are called convex invariants, denoted by, h,c,r and e respectively.

Definition 1.19. A convexity space X is said to be *join hull commutative* (JHC) if for any convex set C and any $p \in X$,

$$Co(C \cup \{p\}) = \cup\{Co(\{c,p\}) : c \in C\}.$$

Definition 1.20. An interval convexity space X is said to have the

1. *Pasch property* if for any a,b,p of X, $a' \in I(a,p)$ and $b' \in I(b,p)$ implies that $I(a,b') \cap I(a',b) \neq \phi$.

2. *Peano property* if for any a,b,c,u,v of X such that $u \in I(a,b)$, $v \in I(c,u)$, there is a v' in I(b,c) such that $v \in I(a,v')$.

If X is having both the properties it is called *Pasch-Peano space* (PP space).

Definition 1.21. Let V be a vector space over R. Let $\mathscr{F}$ be a nonempty family of a linear functionals on V . Then, $\mathscr{S} = \{f^{-1}(-\infty,a]:F \in \mathscr{F}\}$ generates a convexity $\mathscr{C}$ on V called the *H convexity* generated by $\mathscr{F}$. If $-f \in \mathscr{F}$ whenever $f \in \mathscr{F}$, it is called the symmetric H-convexity.

1.2. BACKGROUND OF THE WORK

Convexity is a very old topic whose origin can be traced back at least to Archimedes. This extremely simple and natural notion was however systematically studied by Minkowski during 1911. Bonnesen and Fenchel [1], Valentine [11] and many others also discuss the early development of the theory.

Among the different aspects of convex analysis, such as quantitative, qualitative and combinatorial, our concern will be the last one, where in the classical theorems of convexity in R^n of combinational type play a significant role.

It is well known that, a subset A of a real vector space is convex if and only if it contains with each pair x and y of its points, the entire line segment joining them. It immediately follows that the intersection of any family of convex sets is again a convex set, though the intersection may be empty. The classical theorem due to Edward Helly (1913) sets the condition under which this intersection cannot be empty. Helly's theorem and the theorems due to Caratheodory (1907) and Radon (1921) made a tremendous impact in the development of combinatorial convexity theory and has been studied, applied and generalised by many other authors [21], [31], [72], [74] since 1950s. These theorems in R^n states as follows [8].

Helly's theorem: Let $B = \{B_1, B_2, \ldots, B_r\}$ be a family of r convex sets in R^n with $r \geq n+1$. If every subfamily of n+1 sets in B has a nonempty intersection then $\bigcap_{i=1}^{r} B \neq \phi$.

Caratheodory's theorem: If S is a nonempty subset of R^n, then every x in the convex hull of S can be expressed as a convex combination of n+1 or fewer points.

Radons theorem: Let $S = \{x_1, x_2, \ldots, x_r\}$ be any set of finite points in R^n. If $r \geq n+2$, then S can be partitioned in to two disjoint subsets S_1 and S_2 such that $Co(S_1) \cap Co(S_2) \neq \phi$.

Not only to generalise these classical theorems of R^n, but also to unify the properties of a variety of mathematical structures such as vector spaces, ordered sets, lattices, metric spaces and graphs, an axiomatic foundation of convexity was laid down by Levi[51].

Let $(X, \mathscr{C})$ be a 'Convexity Space'(convex structure, aligned space, algebraic closure systems [31]). The members of $\mathscr{C}$ are called convex sets and $Co(A) = \cap \{C: A \subseteq C \in \mathscr{C}\}$, the convex hull of A. CO(F), with F finite is called a polytope. A polytope which can be spanned by n or less points (where $n > 0$) will be refered to as an n-polytope. The empty set is a 0-polytope. A 2-polytope $Co(\{a,b\})$ is also called a segment joining a and b. Aconvex structure(or, its convexity) is of arity $\leq$ n provided its convex sets are precisely the sets C with the property that $Co(F) \subseteq C$ for each subset F with cardinality atmost

n. That is, a convexity of arity n is "determined by its n-polytopes".

The standard convexity of a vector space, the order convexity of a poset, convexity in a lattice, semilattice and the convexity in a metric space [12] are examples of convexity spaces of arity 2. The study of H-convexity in a real vector space has been made in [19] and [20].

For a convexity space X there exists four numbers $h(x)$, $c(X)$, $r(X)$, $e(X) \in \{0,1,2,...\}$ called the Helly number, the Caratheodory number, the Radon number and the exchange number (Sierksma number), See Definition 1.18. It may be noted that many authors define the Radon number to be one unit larger, which is defined as the first n such that each set with at least n points has a Radon partition. However, we prefer the Definition 1.18.

Let f be a function defined on the class of all convex structures, and ranging into the set $\{0,1,2,...\}$. Then f is called a convex invariant provided that isomorphic convex structures have equal f-values. Obviously, each of the above defined functions h,c,r,e is a convex invariant. Such functions allow for a classification of convex structures according to their combinatorial properties. The function h,c,r go back to traditional topics in the combinatorial geometry of Euclidean space, and they are therefore called classical convex invariants. Attempts to find the interrelation between these invariants were made by Levi [51], Sierksma [71] and Jamison [45]. We shall mention some of these important results.

Levi's theorem [51]. Let $(X,\mathcal{C})$ be a convex structure. Then the existence of r implies the existence of h and $h \leq r$.

Eckhoff-Jamison inequality [45]. If c and h exists for a convexity space, then r exists and $r \leq c(h-1)+1$ if $h \neq 1$, or $c < \infty$.

Sierksma's theorem [71]. $e-1 \leq c \leq \max\{h,e-1\}$.

There are many other inequalities between these invariants. The different cases regarding the existence or otherwise of c,h,r and e is analysed in [12]. Kay and Womble [46] has shown that Levi's theorem is the only one possible if we assume the finiteness of exactly one of the numbers. Study of generalized Helly and Radon numbers [48],[49], extension of Radon theorem due to Tverberg [74], etc. are also found in literature.

The survey paper by Danzer et al. [31], has considerably stimulated the investigations on various aspects of convexity spaces. In the pioneering paper of Ellis [35]., the condition of join hull commutativity (JHC) was considered though the term was introduced by Kay and Womble [46]. It is known [12] that a JHC space is of arity ≤ 2. Products of convexity spaces were studied by Sierksma [70] and proved that JHC property is productive.

The concept of half space familiar in vector space has been generalized to a convexity space [42]. Four separation axioms (Definition 1.16) were introduced by Kay and Womble [46] and Jamison [42]. Under the assumption of S_1, it is an easy observation that $S_4 \Longrightarrow S_3 \Longrightarrow S_2$.

It is known that, a convex structure is S_3 if and only if it is generated by half spaces and that a lattice is S_4 if and only if it is distributive.

We shall now consider the important concept of interval operators (Definition 1.6) introduced by Calder [22] in 1971 which provide a natural method of constructing convex structures. The segment operator of a convex structure $(u,v) \longrightarrow Co\{u,v\}$ is an interval operator.

Conversely, if I is an interval operator, define a subset C of X to be interval convex provided $I(x,y) \subseteq C$ for all x,y in C, we get a convexity space, called the interval convexity space. If Co denotes the segment operator of $\mathscr{C}$, then for any a,b in X, $I(a,b) \subseteq Co\,\{a,b\}$. The two operators need not be equal. It is an important observation that, though the standard intervals and order intervals are convex, the metric interval $\{z \in X:d(x,z)+d(z,y) = d(x,y)\}$ [52] need not be convex. Also, a convexity space is induced by an interval operator if and only if it is of arity $\leq$ 2. Another important property of interval convexity which is of interest to us is Pasch-Peano property (Definition 1.20). These properties are known to hold for vector spaces. Some interesting results in this direction are,

Theorem 1.1 [22]. A convexity space of arity two is JHC if and only if its segment operator satisfies the Peano property.

Theorem 1.2 [35]. A convexity space of arity two is S_4 if and only if the segment operator of X has the Pasch property.

Another interesting concept is that of starshapedness (Definition 1.17). It was proved by Brunn in 1913 [47] that for R^n with standard convexity, the star center of each set is convex.

Several other aspects of convexity theory has been studied by many authors. The prominent among them include the theory of convex geometries [34], ramification property due to Calder [22] and Bean [17], Prenowits [9] theory of join spaces linking up with the theory of ordered geometry, the theory Bryant-Webster spaces [21] and Eckoff's partition conjecture [45].

Since 1950s the theory of convexity spaces has branched and grown into several related theories. An elegant survey has been done by Van de vel [12] whose work has been acclaimed as remarkable.

Attempts were also made by Changat, M and Vijayakumar, A [28] to evaluate the convex invariants of order and metric convexities of Z^n and Onn [58] has studied the Radon number of integer lattice.

Regarding the application part of convexity theory, interesting problems attempted include the determination of computational complexity of the construction of convex hulls and computational complexity of the evaluation of convex invariants. A bibliography on digital and computational convexity has been prepared by Ronse [68].

CONVEXITY IN GRAPHS

It is natural that the concept of convexity could be introduced in graphs also, via its intrinsic metric. Convexity problems in graphs is an emerging line of research in metric graph theory and has proved to be quite successful with respect to applications also, such as facility location problems, dynamic researching in graphs etc. [54]. Several convexities can be defined in a graph, most widely discussed being the geodesic convexity [73] and the minimal path convexity [33] (Definition 1.7). It is obvious that any m-convex set is d-convex. Introducing the notion of an interval function of a graph, Mulder [53] observed that geodesic interval in a graph need not be convex. He called a graph to be interval monotone if all its intervals are convex.

Edelman and Jamison [34] studied the convexity spaces satisfying the antiexchange law (Definition 1.15) and are called the convex geometries or antimatroids. It was observed that antimatroids are precisely convex structures satisfying the Krein-Milman property that, every convex set is the convex hull of its extreme points. They investigated this property for graphs also and proved that,

Theorem 1.3 [38]. G is chordal if and only if the minimal path of convexity is a convex geometry.

Theorem 1.4 [38]. G is a disjoint union of Ptolemaic graphs if and only if the geodesic convexity is a convex geometry.

Theorem 1.5 [44]. G is a connected block graph if and only if the connected alignment is a convex geometry.

Bandelt [14] studied separation properties in graphs and Chepoi [29] gave a characterization of S_3, S_4 and JHC in a bipartite graphs. The geodesic convexity and the m-convexity being defined in terms of intervals, they have some interesting properties.

Theorem 1.6 [12]. A connected graph with Pasch property is interval monotone.

Theorem 1.7 [12]. If a connected graph is S_3 with respect to geodesic convexity, then it is interval monotone.

Theorem 1.8 [12]. Ptolemaic graphs with respect to geodesic convexity are interval monotone.

Theorem 1.9 [14]. The geodesic convexity of a bipartite graph G is S_3 if and only if G embeds isometrically in a hypercube.

Considerable attempts have been made by Bandelt [14], [15], Duchet [32] and Farber-Jamison [38] to evaluate the convexity parameters in graphs. Some interesting results in this context are,

Theorem 1.10 [32] Caratheodory number of any graph with respect to m-convexity is atmost 2.

Theorem 1.11 [33]. Let G = (V,E) be a connected graph with at least two vertices and suppose the maximum size of a clique in G is ω. Denote by h(G) and r(G) respectively the Helly number and the Radon number of the minimal path convexity of G. Then

$$h(G) = \omega$$

$$r(G) = \omega + 1, \text{if } \omega \geq 3$$

$$r(G) = 4 \quad , \text{if } \omega \leq 2$$

It is also proved that the Radon number of the minimal path convexity in a triangle free graph G is 3 if and only if the block graph of G is a path. It is known that the Helly number of a graph with respect to d-convexity is bounded from below by ω (G). Generalizing the results for chordal graphs and distance hereditary graphs due to Chepoi [29], Duchet [32] and others, Bandelt and Mulder [15] proved that $h(G) = \omega(G)$ for a dismantlable graph (Pseudomodular graph). For other related results, see [26] [36] [37] and [69].

As an attempt towards the classification of graphs according to the number of nontrivial convex sets, considerable study has been made by Hebbare [13],[39], [41], Rao and Hebbare [66] and Batten [16]. They called, the empty set, singletons, vertices inducing a complete subgraph and V(G) to be trivial convex sets. A graph is called (k,ω)-convex if it has exactly k nontrivial convex sets and has clique number ω. The (0,2) convex graphs with respect to the geodesic convexity were called distance convex simple (d.c.s) graphs [41] and such graphs with respect to m-convexity were called m-convex simple (m.c.s) graphs by Changat, M [26]. It is easy to observe that every d.c.s graph of order $p \geq 4$ is a triangle free block. When k=1, (k,w)-convex graphs are called uniconvex graphs [40]. Several other interesting results on planar d.c.s graph, o-convex graphs, (0,3) convex graphs, (1,2) convex graphs are in [41]. Changat, M [26] while studying m-convex simple graphs, has proved that, a connected graph $G \ncong P_3$, having no nontrivial cliques is m.c.s if and only if G is m-self centroidal. Also, a connected graph G is m.c.s. if and only if G has no nontrivial cliques or clique separator. In [27] he has proved that a graph G has geodesic iteration number 1 if and only if G is interval monotone which has Caratheodory number 2. Also, a graph G is interval monotone with respect to m-convexity if and only if the minimal path iteration number of G, min(G) is 1. Some other results are in [24] and [25].

We have thus given a survey of results on the theory of convexity spaces and convexity in graphs, related to the results mentioned in this thesis.

1.3 GIST OF THE THESIS

This thesis consists of five chapters including this introductory one, where in we have given some basic definitions and a survey of results on the theory of abstract convexity spaces and convexity in graphs.

In the second chapter, we study the properties of convex simple graphs, interval monotone graphs and totally non interval monotone graphs. It is observed that, two necessary conditions given by Hebbare [41] are not sufficient. Some of the important observations included in this chapter are,

1. It is obvious that d.c.s. graphs are triangle free and t.n.i.m. But, the converse is not true. We have given two different methods of constructing a triangle free t.n.i.m. graph having exactly k non trivial convex sets.

2. Regarding the separation properties of d.c.s and t.n.i.m graphs, it is found that they are half space free.

3. For d.c.s graph, the convex invariants are,

$h(G) = c(G) = r(G) = 2$ and $e(G) = 3$.

4. Chordal graphs with m-convexity has Brunn's property, though it is not true in general.

5. There is no uniconvex graphs with respect to m-convexity.

6 In difference with the observation mentioned in 1, with m-convexity, for a triangle free, 2-connected graph to be k-convex, it is necessary that there is an 'n' such that $(n-1)(n+2)/2 \le k \le 2^{n}-2$.

7. For any graph with geodesic convexity,if its geodetic iteration number is 1 then it is interval monotone and JHC. Converse need not be true. But, if G is a JHC, interval monotone graph, we can give a bound for gin(S) for $S \subset V(G)$. In fact, $gin(S) \le k$ where $k-1 < \frac{\log |S|}{\log 2} \le k$.

8. If G is a geodetic, JHC graph then gin(G) = 1

The third chapter deals mainly with the concept of *solvable trees*, which was introduced to answer the problem, of finding the smallest d.c.s. graph containing a given tree of order atleast four. We say that a tree T is solvable if there is a planar d.c.s. graph G such that T is isomorphic to a spanning tree of G. We prove that,

9. Any tree of order atmost nine is solvable. The bound for the order is sharp. We note that there are graphs of

order 10 which are not solvable.

10. Trees of diameter three, five and trees of diameter four whose central vertex has even degree are solvable. There are trees of diameter six which are not solvable.

A similar problem was posed, with respect to m-convex simple graphs and found that,

11. The size of the smallest m-convex simple graph containing a tree T satisfies, $p-1+m/2 \leq q \leq p+m-2$ where $p = |V(T)|$ and m is the number of pendent vertices of T.

We further study the convexity properties of product of graphs and have,

12. If G_1 and G_2 are d.c.s. graphs then $G_1 \times G_2$ is not so.

13. If G_1 and G_2 are connected, triangle free graphs, $G_i \not\cong K_1$ or K_2 for i = 1,2, then $G_1 \times G_2$ is m-convex simple.

14. If G_1 is m.c.s. and G_2 is any triangle free graph, then $G_1 \times G_2$ is m.c.s.

We conclude this chapter with a discussion on the centers of d.c.s. graphs.

15. If G is a planar d.c.s. graph, then G is self centered if diam (G) = 2 and diam(G) = 2rad(G) or 2rad(G)-1, if diam(G) > 2, C(G) is isomorphic to $\bar{K}_2$ or C_4 according as diam(G) = 2rad(G) or 2rad(G)-1.

In the fourth chapter, we initiate the study of convexity for the edge set of a graph, which is less studied earlier. We define $S \subset E(G)$ to be cyclically convex if it contains all edges comprising a cycle whenever it contains all but one edges of this cycle. This convexity space $(G,\mathscr{C})$ satisfies the exchange law also and hence is a matroid. Further,

16. The arity of $(G,\mathscr{C})$ is 1 if G is a tree and is one less than the size of the largest chordless cycle in G, otherwise.

Thus, $(G,\mathscr{C})$ is not an interval convexity space in general. The convex invariants have also been evaluated.

17. If G is a connected graph of order p, Helly number h(G) = p-1.

18. Caratheodory number $C(G) = 1$ if G is a tree

$= \mathrm{circ}(G)-1$, otherwise.

19. Radon number of $(G,\mathscr{C})$, $r(G) = p-1$.

20 For a connected graph G, the exchange number,

$e(G) = 2$ if G is a tree or a cycle.

$= \max \{\mathrm{circ}(G-v)/v \in V(G)\}$, otherwise.

By generalizing the Pasch-Peano properties to any convexity space, we have obtained a forbidden subgraph characterization also.

21. The convexity space $(G,\mathscr{C})$ is a Pasch space if and only if K_4-x is not an induced subgraph of G.

22 The convexity space $(G,\mathscr{C})$ is a Peano space if and only if G does not contain K_4-x as a subgraph.

Though, for a matroid the Peano property implies Pasch, the converse need not be true by the observations made above.

The last chapter deals with some problems on the H-convexity of R^n. The motivation for this study is the problem posed in [12]. A symmetrically generated H-convexity need not be JHC or S_4. Van de Vel asked as to whether each symmetric H-convexity of R^n $(n>2)$ is of arity two ? We have obtained

23. The arity of the H-convexity in R^3 symmetrically generated by a family of linear functionals corresponding to a family of planes intersecting in a line, is two.

24. An example of an H-convexity in R^3 of infinite arity.

25. The H-convexity symmetrically generated by a family $\mathcal{F}$ of linear functionals from $R^3 \longrightarrow R$, is S_4 if and only if for any two intersecting convex straight lines, the plane determined by these lines is convex.

26. An example of an H-convexity which is neither JHC nor S_4 but is Pasch and Peano and hence not of arity two.

The study initiated in thesis is definitely far from being complete. The last section of this chapter is a list of problems that remains to be tackled, which include some interesting problems posed by others also.

We have included as an appendix, a counter example to a conjecture of Chang [23] on the centers of chordal graphs.