CHAPTER II

II

CONVEX SIMPLE GRAPHS AND INTERVAL MONOTONICITY

In this chapter, we focus on the properties of convex simple graphs. Though any distance convex simple graph is totally non interval monotone, the converse is not true. We give two methods of constructing a triangle free t.n.i.m. graph having exactly k non trivial convex sets. It is also observed that d.c.s graphs and t.n.i.m. graphs are halfspace free. However, with respect to minimal path convexity it is seen that there are no uniconvex graphs and that, values of k for which a k-convex graph exists should satisfy certain conditions. We further concentrate on the iteration number of an interval monotone, JHC graph and also a geodesic, JHC graph.

2.1 DISTANCE CONVEX SIMPLE GRAPHS AND TOTALLY NON INTERVAL MONOTONE GRAPHS

Let us first consider the two necessary conditions for a graph G of order at least five to be distance convex simple.

Theorem 2.1 [41]. A d.c.s graph G of order at least five satisfies the following conditions.

C1. For any 2-path u-v-w in G, there is an x in V such that < {u,v,w,x} > is a chordless 4-cycle of G.

C2. For any 4-cycle u-v-w-x-u in G there is a y in G such that y is adjacent to either u and w or v and x.

Q_3-graph of the 3-cube satisfies C1 but is not d.c.s. We first observe that C1 and C2 are not sufficient conditions. The graph in Fig.2.1 satisfies both the conditions but is not d.c.s.

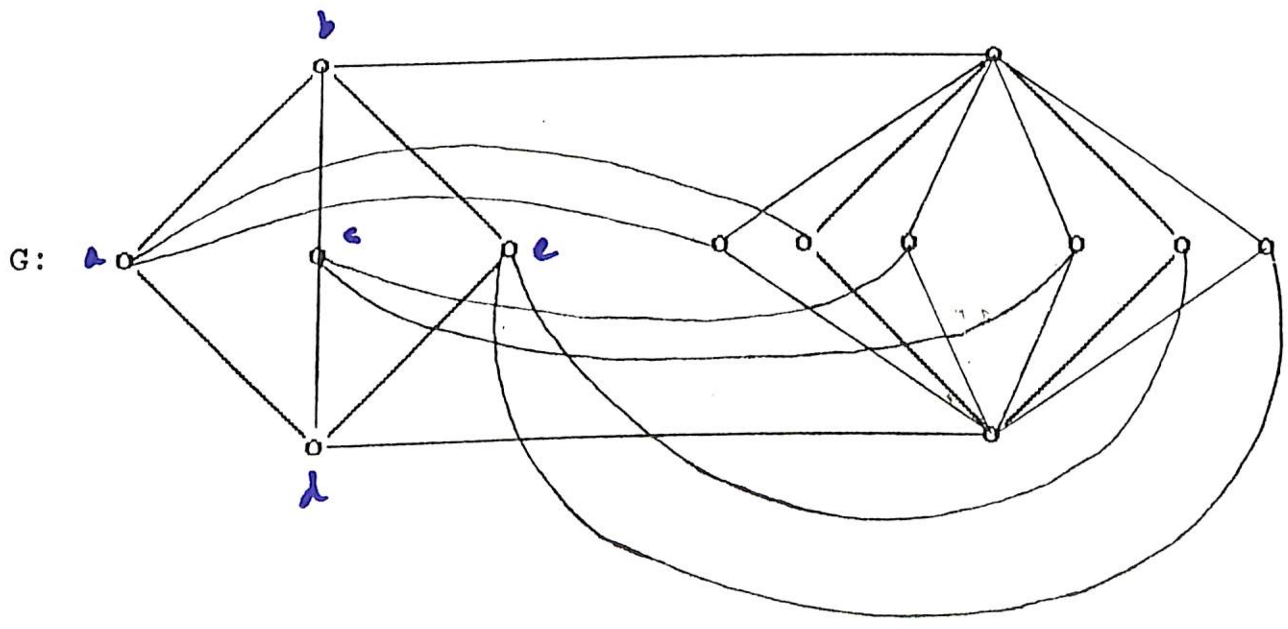


Fig. 2.1

In G, {a,b,c,d,e} is a convex set.

All connected graphs of order atmost three, $K_{m,n}$ for m,n > 1. $\bar{K}_{n_1}+\bar{K}_{n_2}+\ldots+\bar{K}_{n_r}$, $n_i > 2$ for i = 1,2,...,r are examples of d.c.s graphs.

The following theorem gives another class of d.c.s graphs.

Theorem 2.2. [13] Let G be a triangle free graph. Then the graph $D_\lambda(G)$ obtained by taking λ copies, $G_1, G_2, \ldots, G_\lambda$ of G and joining each vertex u_i in G_i to the neighbours of the corresponding vertex u_j in G_j for i,j = 1,2,...,λ, is a d.c.s graph for $\lambda > 1$.

The graph $D_2(C_5)$ is shown in Fig. 2.2.

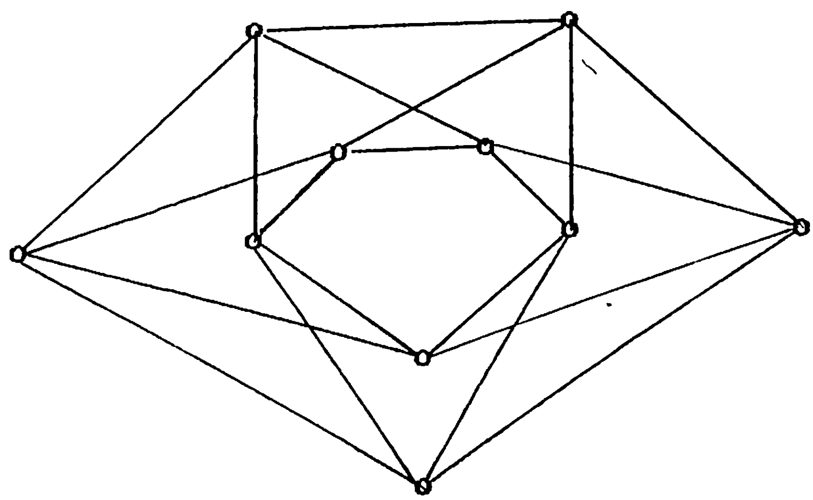


Fig. 2.2

The following theorems from [41] are of much use to us.

Theorem 2.3. Let G be a planar connected graph of order at least four. Then the following are equivalent.

1. G is d.c.s.

2. G is a block without an induced subgraph isomorphic to a cycle C_3, C_n for $n > 4$ or a 6-cycle with exactly one bichord.

3. For each vertex u of degree at least three, there is a unique vertex u' in G such that N(u) = N(u').

Two such vertices u and u' are called partners.

Theorem 2.4. A d.c.s graph G(p,q) is planar if and only if $q = 2p-4$.

Theorem 2.5. [66] Let G be a connected, planar graph of order $p \geq 4$ and $G \not\cong Q_3$. Then G is a d.c.s graph if and only if it satisfies C1.

Interval monotone graphs [53] are those for which all its intervals are convex. Trees, hypercubes, Ptolemaic graphs are examples of interval monotone graphs. A graph is totally noninterval monotone (t.n.i.m) if no nontrivial geodesic interval is convex. It is clear that I(a,b) is convex whenever a = b, a adjacent to b or I(a,b) = V(G). These are called the trivial geodesic intervals.

Note 2.1. A t.n.i.m. graph satisfies the conditions C1 and C2. Otherwise, if u-v-w is a 2-path in G such that there is not an x adjacent to u and w, then I(u,w) = {u,v,w} will be a convex interval. Similarly, if C2 is not satisfied, then the cycle u-v-w-x-u gives the convex interval

I(u,w) = {u,v,w,x}.

However, the conditions C1 and C2 are not sufficient for a graph to be t.n.i.m. In the graph of Fig. 2.1, I(a,e) = {a,b,c,d,e} is a convex interval.

It is clear that d.c.s graphs are triangle free and t.n.i.m. But the converse is not true. The graph G of Fig.2.3 is a triangle free t.n.i.m. graph which is not d.c.s.

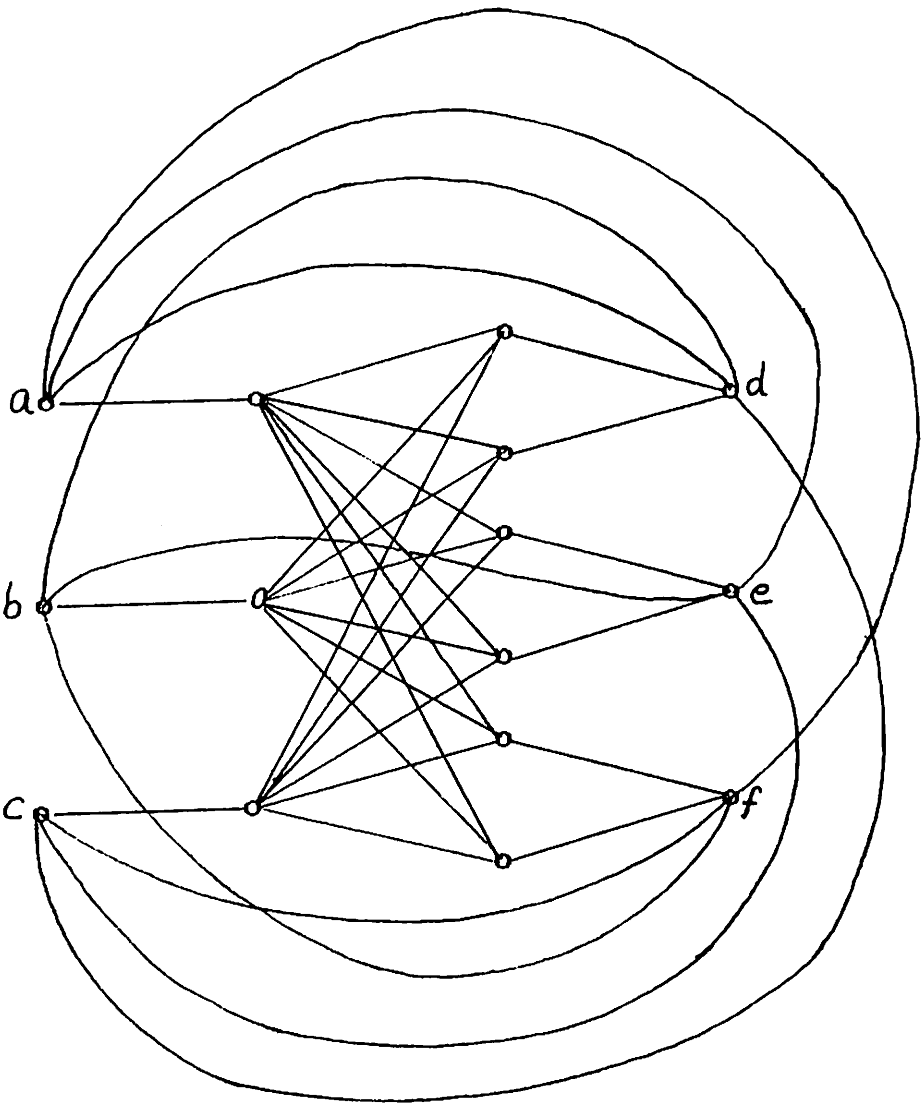


Fig 2.3

In G, the only nontrivial convex set is {a,b,c,d,e,f} and it is not an interval. That is, G is a uniconvex graph in which no nontrivial interval is convex.

Since any connected graph of order atmost five which satisfies C1 and C2 can be expressed as an interval

(C_4 and $K_{2,3}$, which are the only such graphs, can be expressed as interval) a convex set in a trianglefree t.n.i.m. graph will contain at least six vertices.

However, for a triangle free planar graph G, the following theorem holds.

Theorem 2.6. Let G be a triangle free planar graph. Then G is d.c.s if and only if it is t.n.i.m.

Proof: If G is d.c.s then it is t.n.i.m trivially. Now, let it be a triangle free planar t.n.i.m graph. Then $G \neq Q_3$ (the 3-cube) because Q_3 is not t.n.i.m. Now by theorem 2.4, G is d.c.s..

We shall now give two methods of constructing a triangle free, t.n.i.m graph, having exactly k non trivial convex sets.

CONSTRUCTION 1. Let G be a d.c.s graph with $I(a,b) \neq V(G)$ for any $a,b \in V(G)$ and let G_1, G_2 and G_3 be three copies of G. Join each vertex of G_1 to the corresponding vertices of G_2 and G_3 and each vertex of G_2 to the neighbours of corresponding vertices of G_3. The resulting graph is denoted by G^1.

Remark 2.1 G^1 can also be obtained by taking $K_2 \times G$ and then multiplying all the vertices of the copy of G corresponding to one of the vertices of K_2. Also if $u.v \in G$ and u_i, v_i are the vertices corresponding to u and v, for i = 1,2,3. Then

$$d(u_i, v_i) = d(u,v) \text{ for } i = 1,2,3$$

$$d(u_1, v_2) = d(u_1, v_3) = d(u_1, v_1) + 1$$

The graphs induced by $G_1 \cup G_2$ and $G_1 \cup G_3$ are isomorphic to $G \times K_2$ and that induced by $G_2 \cup G_3$ is $D_2(G)$.

Claim: G^1 is having exactly one convex set and it is $V(G_1)$.

It is enough to prove that $Co(\{u,v\}) = V(G_1)$ whenever $u,v \in V(G_1)$ and $Co(\{u,v\}) = V(G^1)$ if one of u and v is in G_2 or G_3.

Case 1: Let $u_1, v_1 \in V(G_1)$ be non adjacent vertices. Let $w \in G$. Then $d(u_1, w_2) = d(u_1, w_1) + 1$ where w_i is the corresponding vertex of w in G_i for i=1,2,3.

Also $d(v_1, w_2) = d(v_1, w_1)+1$. Hence,

$d(u_1,v_1) \leq d(u_1,w_1)+d(v_1,w_1) < d(u_1,w_2)+d(v_1,w_2)+d(v_1,w_2)$. Hence, w_2 is not on a u_1-v_1 shortest path. Now, because G is d.c.s no nontrivial subset of G_1 is convex. Hence, $Co(\{u_1,v_1\}) = V(G_1)$.

Case 2. If $u,v \in G_2 \cup G_3$, then by theorem 1.2, $G_2 \cup G_3$ induce a d.c.s graph and hence $V(G_1)$, $V(G_2) \subset Co\ \{u,v\}$. Now, for any $w \in G$, $w_2,w_3 \in Co\ (\{u,v\})$, where w_2,w_3 are copies of w in $G_2$and G_3. w_1 is on a shortest w_2-w_3 path and hence $w \in Co(\{w_2,w_3\}) \subset Co(\{u.v\})$. Therefore $Co(\{u,v\}) = V(G^1)$.

Case 3. Let $u_1 \in G_1$ and $v_2 \in G_2$ (similarly when $v_3 \in G_3$). Then $u_2,v_1 \in Co(\{u_1,v_2\})$. Now, since $N(u_2) = N(u_3)$, u_3 is on a shortest u_1-v_2 path. That is $u_2;u_3 \in Co(\{u_1,v_2\})$. Then, as in case 2, $Co\ (\{u_1,v_2\}) = V(G^1)$.
Now, since $V(G_1)$ cannot be expressed as an interval, G^1 is t.n.i.m. Taking G^1 in the place of G, construct G^2 in which $V(G_1^1)$ and $V(G_1)$ are the only convex sets. Proceeding like this we get G^k in which $V(G_1)$, $V(G_1^1)$, $V(G_1^2),\ldots,$ $V(G_1^{k-1})$ are the only convex sets.

CONSTRUCTION 2. Let G be a d.c.s graph in which $I(a,b) \neq V(G)$ for any $a,b \in V(G)$. Replace each vertex of a star $K_{1,k}$ by a copy of G. Join each vertex of the copy G_u of G corresponding to the center of $K_{1,k}$ to the corresponding vertices of the other copies. Now, replace each vertex of G_u by a pair of nonadjacent vertices. The graph G so obtained is a triangle free t.n.i.m graph with exactly k convex sets.

Remark 2.2. In general, the k-convex graphs obtained by Construction 1 and Construction 2 are not isomorphic. In Construction 1 the convex sets of G^k form an ascending chain $V(G_1) \subset V(G_1^1) \subset \ldots \subset V(G_1^{k-1})$. But in Construction 2, the k convex sets are disjoint. However, when k=1 both the constructions give the same graph.

We shall now discuss the separation properties (Definition 1.16) of d.c.s graphs. Any graph trivially satisfies S_1 property. The graphs in Fig.2.4 indicate that there are graphs satisfying S_i but not S_{i+1}, for i = 1,2,3.

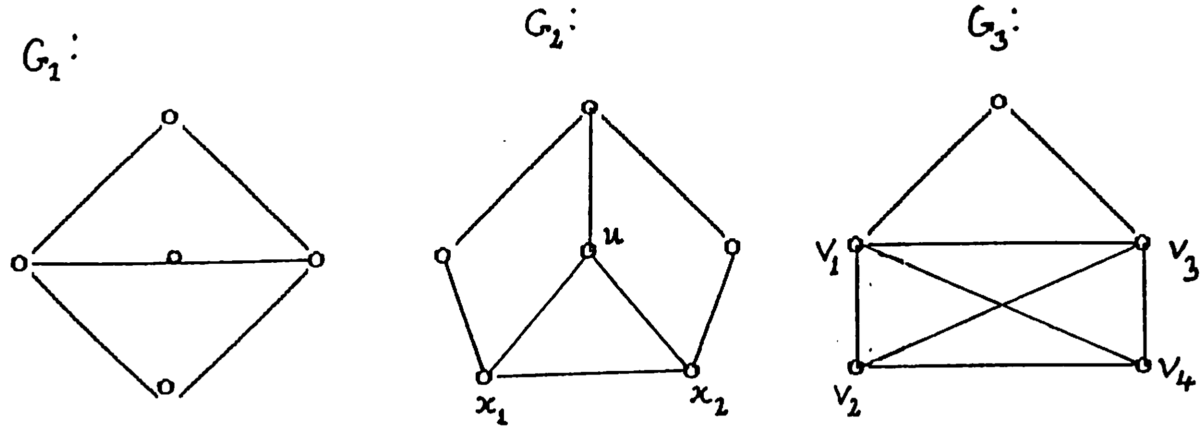


Fig 2.4

G_1 is not S_2 . G_2 is S_2 but not S_3 . Here, there is no halfspace separating the convex set $\{x_1, x_2\}$ and the vertex u. G_3 is S_3 but not S_4 . The convex sets $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are disjoint convex sets which cannot be separated by halfspaces.

There are graphs for which $V(G)\backslash C$ is not convex for any convex set C. We make the following.

Definition 2.1. A graph G is halfspace free if no subset of V(G) is a halfspace.

Theorem 2.7. A connected triangle free graph G of order at least five is halfspace free if it satisfies the conditions C1 and C2.

Proof. Let G be a connected triangle free graph satisfying C1 and C2. Let $C \subset V(G)$ be a convex subset. To prove that $V(G)\setminus C$ is not convex.

Let $u \in V(G)\setminus C$, $v \in V(G)\setminus C$ and $uv \in E(G)$.

Let $w \in V(G)$, $w \neq v$ and $wu \in E(G)$. Note that such a vertex exist because G is of order at least five and it satisfies C1. Now w-u-v is a 2-path and by C1 there is an x in V(G) which is adjacent to w and v (see Fig.2.5)

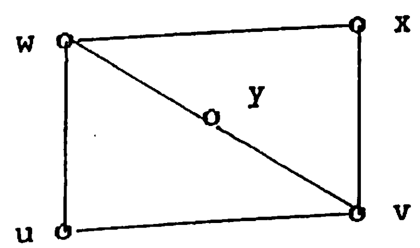


Fig. 2.5.

Now $x \in V(G)\setminus C$ because C is convex, $v \in V(G)\setminus C$ and $v \in Co(\{u,x\})$.

If $w \in V(G)\setminus C$, it is not convex because $u \in Co(\{w,v\})$, but $u \in C$. So let $w \in C$ and $x \in V(G)\setminus C$.

Now, w-u-v-x-w is a 4-cycle in G and by C_2, there is a vertex y adjacent to either w and v or u and x.

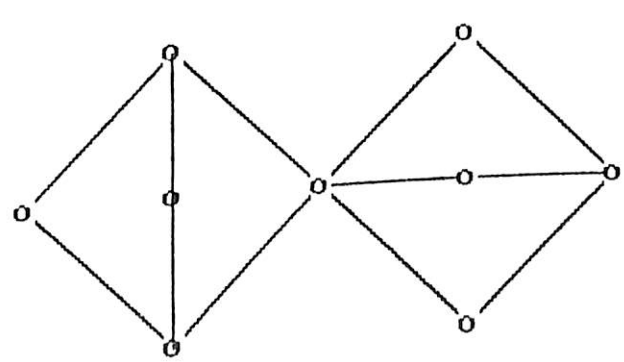
Let y be adjacent to w and v. Then $y \notin C$ because in that case $v \in Co(\{u,y\}) \subset C$ which is a contradiction. Hence, $v \in V(G)\setminus C$. Then, since $w \in Co\{y,x\}$ and $w \in C$, $V(G)\setminus C$ is not convex.

Similar is the case when y is adjacent to u and x. Hence, for any convex set C, $V(G)\setminus C$ is not convex. That is,there is no halfspace in G.

Corollary. Distance convex simple graphs and t.n.i.m graphs are halfspace free.

Note 2.2. Neither C1 nor C2 is necessary for a graph to be halfspace free. The graph G_1 of Fig.2.6 does not satisfy C1, and G_2 of Fig.2.6 does not satisfy C2, but both are halfspace free.

G_1:



G_2:

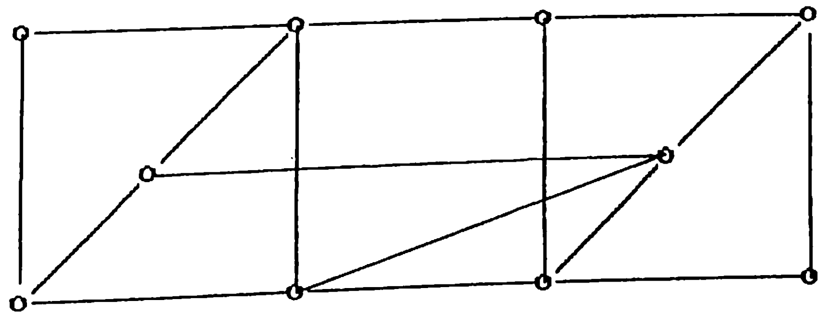


Fig 2.6

The convex invariant could easily be determined for d.c.s graphs. If G is a d.c.s graph, then any set S of three vertices contains a pair u.v of non adjacent vertices and Co({u,v}) = V(G). This observation leads to

Theorem 2.8. For a d.c.s graph G, h(G) = c(G) = r(G) = 2 and e(G) = 3.

It is interesting to observe the star center (Definition 1.17) of a d.c.s graph. It is known that

Theorem 2.9. [12]. A convex structure with Caratheodory number 2 is JHC.

Theorem 2.10 [12]. A JHC convex structure has the Brunn's property.

By theorems 2.8, 2.9 and 2.10 it follows that d.c.s graphs satisfies Brunn's property with respect to the convex hull operator. But when we consider the geodesic interval operator, this will not be true.

For example, the graph G in Fig.2.7 is d.c.s.

G:

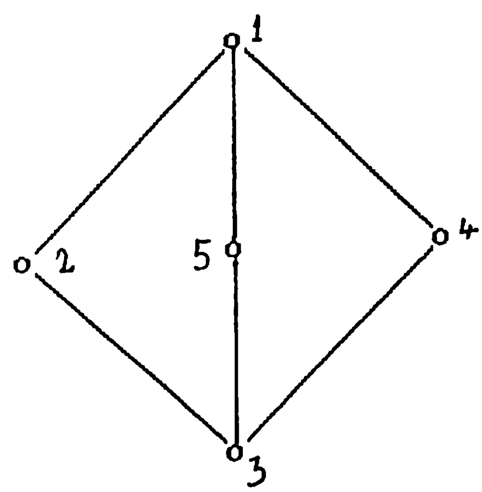


Fig.2.7

Let S = {1,2,3,4}, Ker(S)is the set {2,4} which is disconnected.

2.2. MINIMAL PATH CONVEXITY AND m-CONVEX SIMPLE GRAPHS

In this section by convex sets we mean only m-convex sets and by intervals, only minimal path intervals. It is known (Theorem 1.10) that for any graph G, c(G) is at most 2 and has JHC property. Hence, by theorem 2.10, G has the Brunn's property. But, if Ker(S) is taken with respect to the minimal path interval operator, this is not true. Consider the graph G in Fig.2.8

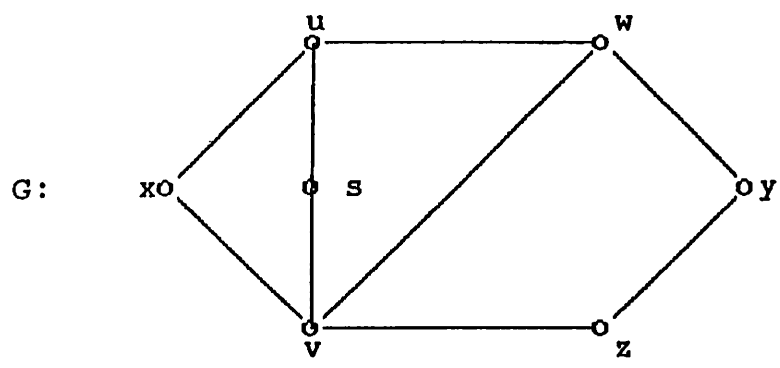


Fig. 2.8

In G, let $S = \{x,u,v,w,z,y\}$. It can be seen that $x,y \in Ker(S)$. But u, which is on a chordless x-y path is not in Ker(S).

However, the following theorem gives a class of graphs for which the Brunn's property holds with respect to minimal path interval.

Theorem 2.11. Let G be a chordal graph and let,

$Ker(S) = \{z \in S: I(z,s) \subset S\}$ for every $s \in S\}$ for $S \subset V(G)$.

Then Ker(S) is convex.

Proof: Let $x, y \in Ker(S)$, and z is on some x-y chordless path where $S \subset V(G)$.

To prove that $I(x,y) \subset Ker(S)$ where,

$I(x,y) = \{z : z$ is on some chordless x-y path$\}$

Since $x, y \in Ker(S)$, $I(x,s) \subset S$, $I(y,s) \subset S$, for every $s \in S$.
Let $z \in I(x,y)$. To prove that $I(z,s) \subset S$ for every s $s \in S$.
Assume without loss of generality that z is adjacent to x.
Let $P_1 = z\text{-}a_1\text{-}a_2\text{-}\ldots\text{-}a_n\text{-}s$ be an z-s chordless path and
$P_2 = x\text{-}z\text{-}b_1\text{-}b_2\text{-}\ldots\text{-}b_k = y$ be an x-y chordless path.
If $x\text{-}z\text{-}a_1\text{-}a_2\text{-}\ldots\text{-}a_n\text{-}s$ is chordless, then clearly
$z, a, \ldots, a_n, s \in S$.

Similarly when $y\text{-}b_k\text{-}\ldots\text{-}b_1\text{-}z\text{-}a_1\text{-}\ldots\text{-}a_n\text{-}z$ is chordless path.
So assume that these are having chords. If ℓ is such that
a_ℓ is adjacent to x, (Note that one end vertex of any chord
of this path is x, because $z\text{-}a_1\text{-}a_2\text{-}\ldots\text{-}a_n\text{-}s$ is chordless).
Then $x\text{-}a_\ell\text{-}a_{\ell-1}\text{-}\ldots\text{-}a_1\text{-}z\text{-}x$ is a cycle in G. If $\ell > 1$ this is
a cycle of length at least four and hence has a chord. Thus
we can see that x is adjacent to a_1. Similarly if b_i is

adjacent to a_m for some m=1, we can see that a_1 adjacent to b_1 (see Fig.2.9).

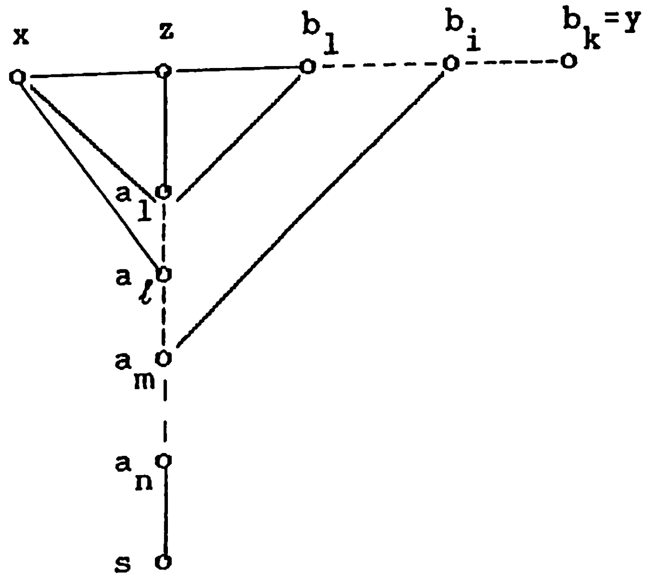


Fig 2.9

Now if $P_3 = x_a - a_1 - \ldots - a_n - s$ is a chordless path or $P_4 = y - b_{k-1} \ldots b_1 - a_1 \ldots a_n - s$ is a chordless path, $a_1, \ldots, a_n \in S$. As above, if x is adjacent to a_ℓ for some $\ell > 1$ then x is adjacent to a_2. Also if b_i is adjacent to a_m for some $m > 1$, a_2 will be adjacent to some vertex on $b_1 - b_2 \ldots b_i$. Let b_j be the first vertex on $b_1 - b_2 - \ldots - b_i$ which is adjacent to a_2. (see Fig.2.10).

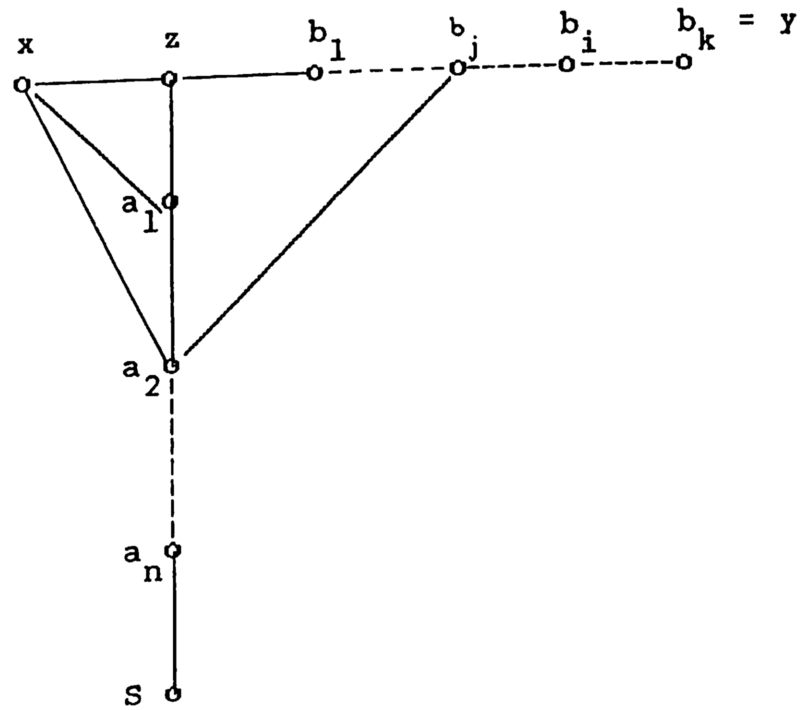


Fig. 2.10.

Then x-z-b_1-b_2-...-b_j-a_2-x will be a chordless cycle of length at least four. Hence either P_3 or P_4 is chordless. Hence $l(z,s) \subset S$ and therefore $z \in Ker(S)$. □

m-convex simple (m.c.s) graphs are those whose only nontrivial convex subsets are the null set, singletons, pairs of adjacent vertices and the whole set $V(G)$. The following theorem gives a necessary and sufficient condition for a graph to be m.c.s.

Theorem 2.12. [26]. A graph is m.c.s if and only if it has no nontrivial clique or clique separator.

It is clear that d.c.s graphs are m.c.s. But the converse is not true. For example, the graph in Fig.2.11 is an m.c.s graph which is not d.c.s.

G:

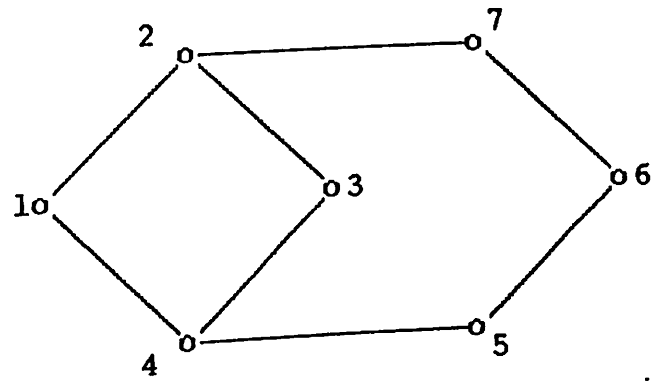


Fig. 2.11

By theorem 2.12 it is clear that G is an m.c.s graph. But it is not d.c.s because {5,6,7} is a nontrivial d-convex set. The question as to whether there exist for any given k, a k-convex graph which is triangle free and totally non interval monotone, with respect to m-convexity also, lead us to following theorems.

Theorem 2.13. There is no uniconvex graph.

Proof: Let G be a graph having a nontrivial convex subset. Then by theorem 2.12, G contains a clique separator S. Let

$C_1, C_2, \ldots, C_n$ be components of $G \setminus S$. Clearly $n \geq 2$. Then $C_i \cup S$ is convex in G because any chordless path connecting vertices of $C_i \cup S$ will be contained in $\langle C_i \cup S \rangle$. Note that since S is complete, any path containing a vertex not in $C_i \cup S$ will have a chord. Thus the number of convex sets is at least two. □

We call a convex set C to be a minimal nontrivial convex subset if no proper subset of C of cardinality at least three is convex.

The following theorem specify the condition on k which is necessary for a graph to be k-convex.

Theorem 2.14. Let G be a k-convex, triangle free, 2-connected graph. Then there is an 'n' such that $(n-1)(n+2)/2 \leq k \leq 2^n-2$.

Proof: Let $C_1, C_2, \ldots, C_n$ be minimal nontrivial subsets of G. Hence $C_i \cap C_j$ contains at most two vertices for $i \neq j$. Otherwise $C_i \cap C_j$ will be a nontrivial convex set which is a proper subset of C_i. Let $C_i \cap C_j = S$ with $|S| = 2$.

Claim 1. S is a clique separator.

Let $S = \{x,y\}$. Then, $Co(S) \subset C_i \cap C_j$. If x is not adjacent to y, $Co(S)$ will be a nontrivial convex subset properly contained in C_i. Hence x is adjacent to y, that is S is a clique.

Now to prove that $G \setminus S$ is disconnected. If not, each pair of vertices in $G \setminus S$ is connected by a path. In particular, each vertex of $C_i \setminus S$ is connected to each vertex of $C_j \setminus S$ by some path in $G \setminus S$. Let $c_i \in C_i \setminus S$ and $c_j \in C_j \setminus S$. Let $c_i - u_1 - u_2 \ldots u_\ell - c_j$ be a chordless $c_i - c_j$ path in $G \setminus S$. Assume without loss of generality that c_i is so chosen that $u_k \notin C_i$, for $k = 1 \ldots \ell$. Since G is triangle free c_i is not adjacent to at least one vertex of S. Let it be x. Consider the path joining c_i and x which contain c_j on it. It is clear that some subset of this will induce a chordless $c_i - x$ path containing a vertex in $C_j \setminus S$. This is not possible because C_i is convex. Hence $G \setminus S$ is disconnected. Therefore, if $C_i \cap C_j = S$, any clique of size at least two, then S is a separator set.

Now, let H be a graph with,

$V(H) = \{C_1, C_2, \ldots, C_n\}$ and C_i is adjacent to C_j, if $C_i \cap C_j$ is a clique separator.

Claim: H is a block graph.

If not there will be a block B in H and $C_i, C_j \in B$ such that C_i is not adjacent to C_j in H. Assume without loss of generality that $d(C_i, C_j) = 2$ and let $i = 1$, $j = 3$. Let C_1-C_2-C_3 be a path and since these are vertices of a block, there will be another path C_3-C_4-...-C_i-C_1 connecting C_3 and C_1.

Let $C_1 \cap C_2 = S_1$, $C_2 \cap C_3 = S_3 \ldots C_i \cap S_1 = S_i$. Note that $S_1 \neq S_2$. Otherwise $S_1 = S_2 \subset C_1 \cap C_3$ and hence C_1 will be adjacent to C_3 which is a contradiction. Now, since G is triangle free, we get an $x \in S_1, y \in S_2$ such that x is not adjacent to y. Note that $x, y \in C_2$. Assume without loss of generality that $S_1 \neq S_i$ and $S_2 \neq S_3$. [If $S_1 = S_i = S_{i-1} = ,,, = S_3$, $C_1 \cap C_2 = S_3$, $C_3 \cap C_4 = S_3$ and C_1 will be adjacent to C_3. Similarly when $S_2 = S_3 = \ldots\ldots = S_i$.]

Now because $C_1 \cap C_i \neq \phi$, $C_i \cap C_{i-1} \neq \phi$, ... $C_3 \cap C_2 \neq \phi$ and $x \in C_1$ and $y \in C_2$, we get an x-y path through C_i, C_{i-1}, ..., C_3, C_2 and hence a chordless path joining x and y containing vertices of these sets. That is C_2 is not convex, which is a contradiction. Hence H is a block graph.

Now observe that if C_1 and C_2 are convex and $C_1 \cap C_2 = S$, a clique separator, $C_1 \cup C_2$ is convex. Hence, the convex sets of G are those corresponding to the connected subsets of H. It is known that the number of connected sets of a block graph is minimum when it is a path and is a maximum when it is a complete graph. The number of connected sets other than the whole set is $(n-1)(n+2)/2$ when it is a path and it is 2^n-2 when it is a complete graph. Hence the number of connected sets in H lies between $(n-1)(n+2)/2$ and 2^n-2. Therefore, G is a k-convex graph implies that there is an n such that $(n-1)(n+2)/2 \leq k \leq 2^n-2$ □.

Illustration: If $n = 1$, then $k = 0$ and G is an m.c.s. graph.

If $n = 2$, then $k=2$. So, there is no uniconvex graph.

If $n = 3$, then $5 \leq k \leq 6$, so there is no 3-convex graph or 4-convex graph.

If $n = 4$, then $9 \leq k \leq 14$, so there is no 7-convex or 8-convex graphs.

Remark 2.3. In the theorem 2.14, for any $C_i \in H$, if $N(C_i)$ consists of m pairwise nonadjacent vertices, then the subgraph of G induced by C_i consists of at least m-edges. This is because if $C_1 \cdots C_n$ are the neighbours of C_i which are pairwise nonadjacent, then in G,

$C_i \cap C_k \neq S_k \simeq K_2$ for $k = 1,\ldots,m$ and $S_k \neq S_\ell$ for $k \neq \ell$

COROLLARY: Let H be a block graph of order p. Then there is a t.n.i.m graph G' such that G' is k-convex where k is the number of convex subsets of H other than the null set and the whole set.

Proof: Let $G \simeq K_{n,n}$ $n \geq 3$. Take n to be sufficiently large so that if $C_1, C_2, \ldots, C_m$ are the vertices of H as in the Remark 2.3, then $m \leq n^2$. Let $V(H) = \{C_1, C_2, \ldots, C_p\}$. Now form G' as follows. Let $G_1, G_2, \ldots, G_p$ be p copies of G. Identify an edge of G_i with the corresponding edge of G_j if and only if C_i is adjacent to C_j in H. That is, $G_i \cap G_j \simeq K_2$ in G' if C_i is adjacent to C_j in H. Now, the nontrivial convex sets of G' are those corresponding to the convex sets of H different from the null set and the whole set.

Now we prove that none of these is an interval.

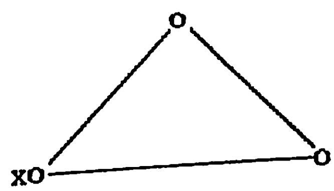
If $a,b \in G_i$ for some i, then, I(a,b) cannot be convex. Assume that $a \in G_1$, $b \in G_2$. Then any path connecting a and b contain the vertices of a clique separator S where $S \subset V(G_1)$. Let V_1 and V_2 be the bipartition of V(G). Let $V_{1,1}$ and $V_{1,2}$ be the corresponding sets in $V(G_1)$. Let $a \in V_{1,1}$ (similarly when $a \in V_{1,2}$). Let $a_1 \in V_{1,1} \setminus (S \cup \{a\})$. Such a vertex exist because $S \cap V_{1,1}$ is a singleton and $|V_{1,1}| \geq 3$.

Claim: $a_1 \notin I(a,b)$.

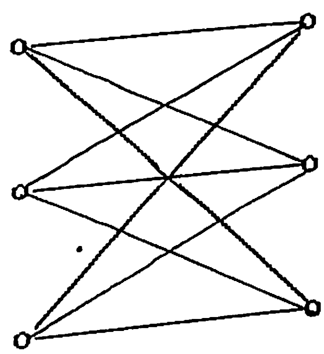
If $a-b_1-a_1-b_2-a_2 \ldots -b$ is an a-b path then $a-b_2$ is a chord. Hence, there does not exist a chordless a-b path containing a_1. Hence no nontrivial interval is convex.

Illustration:

H:



G:



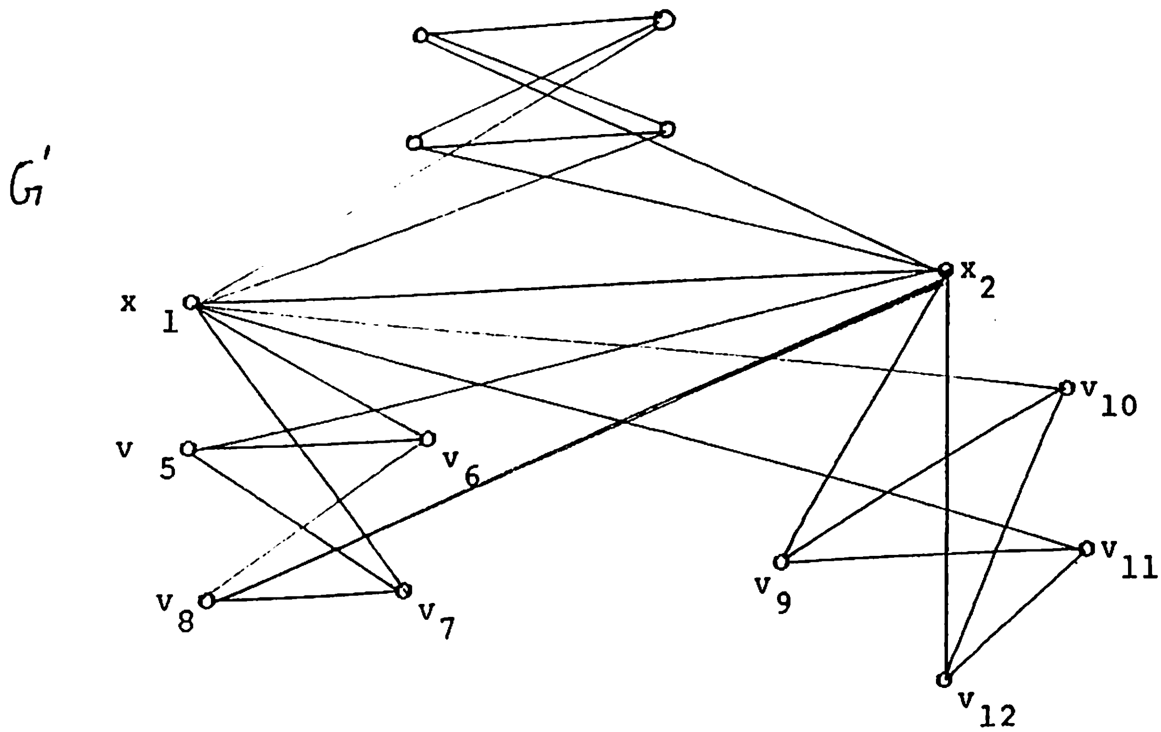


Fig. 2.12

2.3. ITERATION NUMBER

Minimal path iteration number of a graph G, min(G) [27] is a concept analogous to geodetic iteration number (Definition 1.11). It is obtained in a similar manner by replacing the geodetic interval operator by the minimal path interval operator.

It can be observed that for any given k, the sequential join of k+1 copies of $\bar{K}_2$, is a graph which is both d.c.s and m.c.s and both its minimal path iteration number and geodetic iteration number is k.

We know that the Caratheodory number of any graph with m-convexity is atmost 2 and hence JHC. In addition, if G is interval monotone with respect to m-convexity, min(G) = 1 and conversely.

However, in the case of graphs with geodesic convexity it is necessary that G should be interval monotone and JHC in order that gin(G) = 1. But, it is not sufficient (See Fig.2.13).

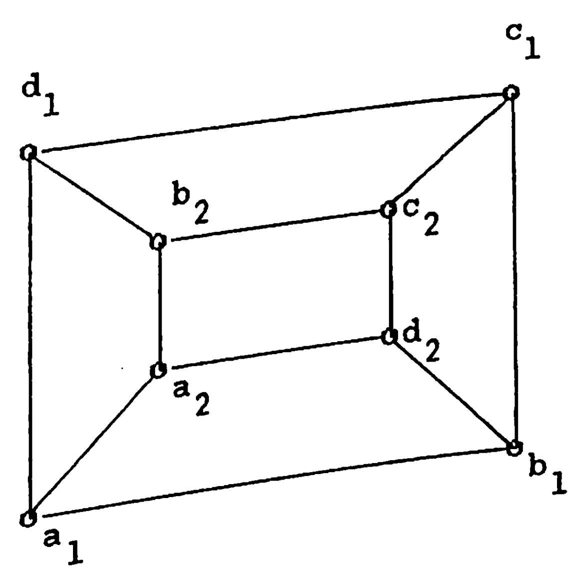


Fig. 2.13.

Let $S = \{a_2, b_1, d_1\}$. Then,

$$S^1 = \{a_1, b_1, c_1, d_1, a_2, b_2, d_2\} \text{ and } S^2 = V(G).$$

Hence gin $(G) \neq 1$.

If G is interval monotone but not JHC, there are graphs G and $S \subset V(G)$ with $|S| = 3$ such that gin(S) is large. However, we have,

Theorem 2.15. Let G be a JHC, interval monotone graph and let $S \subset V(G)$. Then g in $(S) \leq k$, where k is such that

$$k-1 < \frac{\log |S|}{\log 2} \leq k.$$

Proof: Let $S = \{a_1, \ldots, a_n\}$.

Let $C_1 = Co(\{a_1, \ldots, a_{\lceil n/2 \rceil}\})$ and

$C_2 = Co(\{a_{\lceil n/2 \rceil + 1}, \ldots, a_n\})$

Then $Co(S) = CO(C_1 \cup C_2) = \bigcup\{Co(\{c_1, c_2\}) : c_1 \in C_1, c_2 \in C_2\}$, since G is JHC.

$= \bigcup \{I(c_1, c_2) : c_1 \in C_1, c_2 \in C_2\}$ because G is interval monotone.

Hence $Co(S) = \bigcup \{I(c_1, c_2) : c_1, c_2 \in C_1 \cup C_2\}$

$= (C_1 \cup C_2)^1$.

Now let $C_{11} = Co(\{a_1, a_2, \ldots, a_{\lceil n/4 \rceil}\})$

$C_{12} = Co(\{a_{\lceil n/4 \rceil + 1}, \ldots, a_{\lceil n/2 \rceil}\})$

$C_{21} = Co(\{a_{\lceil n/2 \rceil + 1}, \ldots, a_{\lceil 3n/4 \rceil}\})$

$C_{22} = Co(\{a_{\lceil 3n/4 \rceil + 1}, \ldots, a_n\})$

Then $C_1 = Co(C_{11} \cup C_{12})$ and $C_2 = Co(C_{21} \cup C_{22})$. Then as above, $C_1 = (C_{11} \cup C_{12})^1$ and $C_2 = (C_{21} \cup C_{22})^1$

Hence $Co(S) = ((C_{11} \cup C_{12})^1 \cup (C_{21} \cup C_{22})^1)^1$

$= (C_{11} \cup C_{12} \cup C_{21} \cup C_{22})^2$.

$= (Co(\{a_1, \ldots, a_{\lceil n/4 \rceil}\}) \cup Co(\{a_{\lceil n/4 \rceil + 1}, \ldots, a_{\lceil n/2 \rceil}\}) \cup \ldots\}^2$

$= (Co(\{a_1, \ldots, a_{\lceil n/2^2 \rceil}\}) \cup Co(\{a_{\lceil n/2^2 \rceil + 1}, \ldots, a_{\lceil n/2 \rceil}\}) \cup \ldots\}^2$

Proceeding like this,

$$Co(S) = (Co\{a_1,\ldots,a_{\lceil n/2^k\rceil}\} \cup Co\{a_{\lceil n/2^k\rceil+1}\},\ldots a_{\lceil n/2^{k-1}\rceil} \cup \ldots \cup \{a_{\lceil(2^k-1)n/2^k\rceil+1}, a_n\})^k$$

Now, When $\lceil n/2^k\rceil = 1$

$2^{k-1} < n \le 2^k$ and $Co(a_1\ldots a_{\lceil n/2^k\rceil}) = Co(a_1) = \{a_1\}$

$$Co(S) = (\{a_1\} \cup \{a_2\}\ldots \cup \{a_n\})^k$$

$$= (\{a_1 \ \ldots \ a_n\})^k = S^k$$

Hence, gin $S \le k$, where $2^{k-1} < n < 2^k$. That is $k-1 < \frac{\log |S1}{\log 2} \le k$. That is

$k-1 < (\log n/ \log 2) \le k$. □

The following discussion illustrates that there are graphs G and $S \subset V(G)$ such that $gin(S) = k$ where $2^{k-1} < S \le 2^k$.

Let k be any integer and $n = 2^k$. Let Q_n be the n-cube, vertices labelled with (0,1) valued n-tuples.

Let $\delta_i = (x_1,\ldots,x_n)$ where $x_i = 1$ and $x_j = 0$ for $j \ne i$ and $\delta_0 = (0,0,\ldots,0)$. Then, $d(\delta_i,\delta_j) = 2$, for $i \ne 0$.

Let $S = \{\delta_1, \delta_2, \ldots, \delta_n\}$.

If $\delta_{i,j} = (x_1, \ldots, x_n)$ where $x_i = x_j = 1$ and $x_k = 0$ for

$k \neq i,j$, then $\delta_{i,j}$ is adjacent to δ_i and δ_j.

Hence, $S^1 = \{\delta_0\} \cup S \cup N_2(\delta_0)$

Now if $\delta_{i,j}\ \delta_{k,\ell} \in N_2(\delta_0)$ be such that $i,j \neq k,\ell$, then

$d(\delta_{i,j}, \delta_{k,\ell}) = 4$ and if $A = \{i,j,k,\ell\}$ and

$\delta_A = (x_1, \ldots, x_n)$ where $x_i = x_j = x_k = x_\ell = 1$ and $x_m = 0$ for

$m \notin A$, then, $\delta_A \in I(\delta_{i,j}, \delta_{k,l})$.

Hence, $S^2 = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup N_3(\delta_0) \cup N_4(\delta_0)$

$= \{\delta_0\} \cup S \cup N_2(\delta_0) \cup N_3(\delta_0) \cup N_{2^2}(\delta_0)$

Similarly, $S^3 = \{\delta_0\} \cup S \cup \{N_2(\delta_0) \cup \ldots \cup N_{2^3}(\delta_0)$, and

$S_k = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots \cup N_{2^k}(\delta_0) = V(Q_n)$.

Hence, $gin(S) = k$.

Note 2.2. If n is such that $2^{k-1} < n < 2^k$, in the above example,

$$S^{k-1} = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \dots \cup N_{2^{k-1}}(\delta_0)$$

and
$$S^k = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \dots \cup N_{2^{k-1}}(\delta_0) \dots \cup N_n(\delta_0).$$

Therefore if n is such that $2^{k-1} < n \leq 2^k$

$$g\,in\,(S) = gin\,(\{\delta_i\}) = k.$$

If an interval monotone, JHC graph has the additional property that the geodesic intervals are decomposable [12], then g in (G) = 1. Also, we observe that the class of graphs with decomposable intervals are nothing but the class of geodetic graphs. Hence we have,

Theorem 2.16. If G is a geodetic, JHC graph, then gin(G) = 1.

Proof: Since G is geodetic, it is interval monotone. Because G is JHC also, the geodesic interval operator satisfies the Peano property by Theorem 1.3. We denote by ab the shortest path connecting a and b.

Now, let $a,b,c \in V(G)$, $u \in ab$, and $v \in cu$. It is enough to prove that v is in one of the intervals $I(a,b)$, $I(b,c)$ or $I(a,c)$. Because G is geodetic $I(a,b) = ab$.

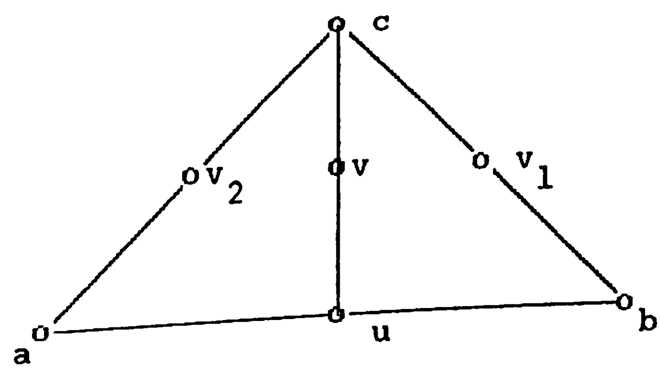


Fig. 2.14.

Assume without loss of generality that $d(c,v) = 1$. Let $d(a,c) = \ell_1$ and $d(b,c) = \ell_2$. Now, by the Peano property, there are vertices $v_1 \in bc$, $v_2 \in ac$ such that $v \in av_1 \cap bv_2$.

Now because $d(a,c) = \ell_1$, $d(a,v) \geq \ell_1 - 1$.

If $d(a,v) = \ell_1 - 1$ then $d(a,c) = d(a,v) + 1 = d(a,v) + d(a,c)$

and hence $v \in ac$.

So assume $d(a,v) \geq \ell_1$.

If $d(a,v) > \ell_1$ then $d(a,v_1) = d(a,v) + d(v,v_1) > \ell_1 + d(v,v_1)$

That is $d(a,v_1) > d(a,c) + d(v,v_1)$

Now $d(a,v_1) \leq d(a,c) + d(c,v_1)$

Therefore $d(v,v_1) < d(c,v_1)$ and

$$\ell_2 - d(c,v_1) < \ell_2 - d(v,v_1)$$

$$d(b,v_1) < \ell_2 - d(v,v_1)$$

$d(b,v_1) + d(v,v_1) < \ell_2$ and so $d(b,v) \leq \ell_2 - 1$ and $d(b,v) < \ell_2 - 1$ is not possible and hence $d(b,v) = \ell_2 - 1$ and in this case $v \in bc$.

Now assume that $d(a,v) = \ell_1$.

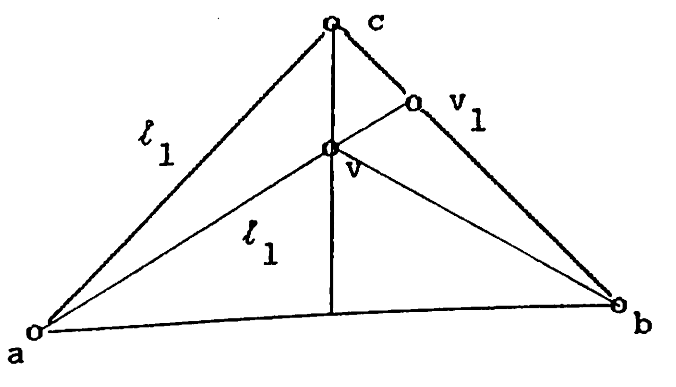


Fig.2.15

In this case $d(a,v_1) = \ell_1 + d(v,v_1)$

Now, if $d(c,v_1) > d(v,v_1)$, then

$$d(b,v_1) + d(v,v_1) \leq \ell_2 - 1 \text{ and hence } v_1 \in bc.$$

So let $d(c,v_1) \leq d(v,v_1)$. But $d(c,v_1) < d(v,v_1)$ is not possible because av_1 is a shortest path containing v.

Therefore $d(c,v_1) = d(v,v_1)$.

But this is again a contradiction because these give two distinct shortest paths connecting a and v_1. □