CHAPTER III

III

CONVEX SIMPLE GRAPHS AND SOLVABILITY

In this chapter, we continue the study of properties of convex simple graphs. Motivated by a problem posed in [41], we define the notion of solvability and make an interesting observation that, all trees of order at most nine are solvable and that the bound is sharp. All trees of diameter three, five, and those with diameter four whose central vertex has even degree are also solvable. However, a characterization of solvable trees is yet to be obtained. A problem of similar type with respect to m-convexity is also discussed. We then discuss about the center of d.c.s graphs. We conclude this chapter with the study of the convexity properties of product of graphs. Some results of this chapter are in [60].

3.1 SOLVABLE TREES

In this section, we introduce the notion of solvable trees associated with a d.c.s graph, to answer the following,

PROBLEM [41] Describe the smallest distance convex simple graph containing a given tree of order at least four.

$K_{2,n}$ is such a graph for $K_{1,n}$. For a tree T which is not a star, let V_1 and V_2 be the bipartition of V(T) with $|V_1|=m, |V_2|=n$, then $K_{m,n}$ is a d.c.s graph containing a tree isomorphic to T. However, to find the smallest d.c.s. graph, we note by theorem 2.4. that, for any d.c.s. graph $q \geq 2p-4$ and the lower bound is attained if and only if it is planar. So, for a given tree T if there exists a planar d.c.s. graph containing T as a spanning subgraph, then that will be the smallest d.c.s. graph containing T. This observation leads us to,

Definition 3.1. A tree T is *solvable* if there is a planar distance convex simple graph G such that T is isomorphic to a spanning tree of G.

From the remarks made above, it is clear that $K_{1,n}$ is not solvable. Hence, in the following discussions we consider only trees which are not stars.

A USEFUL GRAPH OPERATION:

We shall now describe an operation frequently used in this section. Let u and v $\in$ V(G). Join u to all the vertices in N(v) and v to all the vertices in N(u). The resulting graph is denoted by G*(u,v) and in this graph N(u) = N(v).

Remark 3.1 If G is planar and if G can be embedded so that u,v,N(u) and N(v) are all contained in the same face, then G*(u,v) is planar. Also, if u and v are partners then G*(u,v) $\simeq$ G.

Lemma 3.1. Any path of length at least four is solvable.

Proof: Let P be a path of length at least four and let u $\in$ C(P). Then $N_i(u)$ consists of two non-adjacent vertices for i=1,2,...r-1 and $N_r(u)$ is either a pair of non adjacent vertices or a Singleton according as C(P) $\simeq K_1$ or K_2, where r is the radius of P.
Now, the graph G = <u>+<N(u)>+...+<N_r(u)> is a planar d.c.s. graph containing P. □

Theorem 3.2. Any tree of order atmost nine is solvable.

Proof. If T is a path then it is solvable by the lemma 3.1. Suppose that T is not a path. Let u be a vertex of T such that $d(u) \geq 3$ and let $N(u) = \{a_1, a_2, \ldots a_n\}$, $n \geq 3$.

Case I. Any vertex in $N_2(u)$ is of degree one.

Assume that $d(a_1) = \min\{d(a_i) : a_i \in N(u)\}$. Choose $u' \in N_2(u)$ such that $N_2(u) \cap N(a_1) \backslash \{u'\} = \phi$. Construct

$G \simeq T*(u,u')*(a_1,a_2)*\ldots*(a_{n-1},a_n)$ if n is even and

$G \simeq T*(u,u')*(a_2,a_3)*\ldots*(a_{n-1},a_n)$ if n is odd.

Using theorem 2.3 and the remark 2.3, it follows that G is a planar d.c.s. graph which contains T.

Case II. There is a vertex in $N_2(u)$ of degree at least two. Choose $u' \in N_2(u)$ such that $d(u') = \max\{d(v) : V \in N_2(u)\}$ and let $N(u') = \{v_1, v_2, \ldots, v_m\}$. Let $N = N(u) \cup N(u')$. Note that, $m > 3$. Since $|V(T)| \leq 9$, $N(v_i) - \{u,u'\} = \phi$ for at least one value of i.

Sub case 1. $N[u] \cup N[u'] = V(T)$. Then $T*(u,u') \simeq K_{2,p-2}$ is such a planar d.c.s. graph.

Sub case 2. $N[u] \cup N[u'] \neq V(T)$, but

$$N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) = V(T).$$

Without loss of generality assume that $N(v_1)\backslash\{u,u'\} = \phi$.

Then the required graph is $T*<u,u'>*(v_1,v_2)* \ldots *(v_{m-1},v_m)$ if m is even and $T*(u,u')*(v_2,v_3)*\ldots*(v_{m-1},v_m)$ if m is odd.

Sub case 3. $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \neq V(T)$. but

$$N[u] \cup N[u'] \cup [\bigcup_{i=1}^{m} N(v_i)] \cup [\bigcup_{i=1}^{m} N_2(v_i)] = V(T).$$

Here, note that $N(v_i)\backslash\{u,u'\} \neq \phi$ for at most two values of i, say 1 and 2. Let $w_1 \in N(v_1)\backslash \{u,u'\}$ be such that $d(w_1) \geq 2$. Since $|V(T)| \leq 9$, $d(w_1)$ can not exceed three. If $d(w_1) = 3$, by the choice of u', we can see that $w_1 \in N_4(u)$ in T and let $u\text{-}v_2\text{-}u'\text{-}v_1\text{-}w_1$, be the $u\text{-}w_1$ path in T (That is, $v_1 \notin N(u)$ and $v_2 \in N(u')$).

Now, $G \simeq T*(u,w_1)*(v_1,v_2)$ is the required planar d.c.s. graph.

If $d(w_1) = 2$, let $w_2 \in N(w_1) \setminus \{v_1\}$, then

$T*(u,u')*(w_2,v_1)*(v_2,v_3)$ is the required graph.

Sub case 4. $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) \neq V(T)$.

Then,

$$N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) \cup (\bigcup_{i=1}^{m} N_3(v_i)) = V(T).$$

Note that, $N(v_i) \setminus \{u,u'\} \neq \phi$, for only one value of i, there is only one vertex w_1 in it and there are two vertices w_2 and w_3 such that w_1w_2 and $w_2w_3 \in E(T)$.

Then, $T*(u,u')*(v_1,w_2)$ is the required graph. □

Remark 3.2. In theorem 3.2 the upper bound for the order of T is sharp. Consider the tree T of order 10,

T:

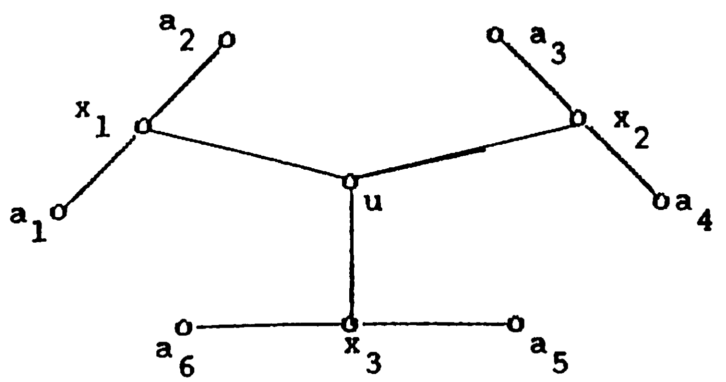


Fig. 3.1

A non- solvable tree of order 10 and diameter 4.

Here, $d(x_i) > 2$ in T and hence also in G. So, by theorem 2.3, for each x_i there is a unique partner x_i' in V(T). Now, $x_i' \neq a_j$ or u because $G^*(x_i,a_j)$ and $G^*(x_i,u)$ will contain a triangle for $i=1,2,3$ and $j=1,2,\ldots,6$. Hence x_i' can only be x_j for some $j \neq i$. Then there will be one x_i for which there is no partner.

Theorem 3.3. The following classes of trees are solvable.

(a) Trees of diameter three.

(b) Trees of diameter four whose central vertex has even degree.

(c) Trees of diameter five.

Proof:(a) Since T is of diameter three, $T \simeq S_{m,n}$ (Definition 1.2.), for $m,n > 0$.

Let c_1 and c_2 be the central vertices and let $N(c_1) = \{a_1,a_2,\ldots,a_m\}$ and $N(c_2) = \{b_1,b_2,\ldots,b_n\}$. Then $T^*(b_1,c_1)^*(a_1,c_2)$ is a planar d.c.s. graph containing T as a spanning tree.

(b) Let $\mathrm{diam}(T) = 4$ and the central vertex c has even degree.

Let $N(c) = \{a_1,a_2,...,a_n\}$ and $c' \in N_2(c)$. Then $T*(c,c')*(a_1,a_2)*...*(a_{n-1},a_n)$ is the required graph.

(c) Let diam(T) = 5. Then T will be as in Fig. 3.2.

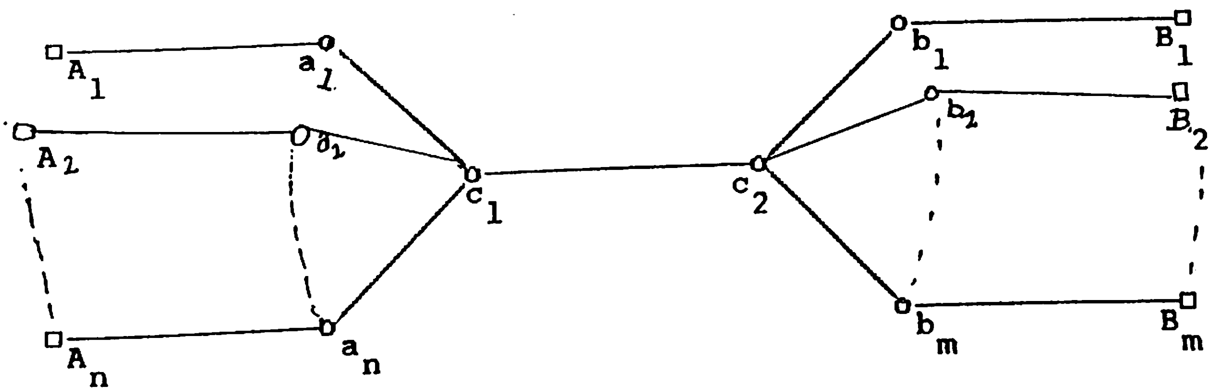


Fig. 3.2

Clearly A_i and B_j are independent sets and are nonemtpy for at least one value each of i and j, i = 1,2,...,n and j=1,2,...,m.

Case 1. Both m and n are even.

Then $\{c_1,a_1,...,a_n\} \cup (\bigcup_{i=1}^{n} A_i)$ and $\{c_2,b_1,...,b_n\} \cup (\bigcup_{j=1}^{m} B_j)$ induce trees say T_1 and T_2 respectively. Note that $diam(T_i) < 5$ for i = 1,2.Choose a c_1' from some A_i and a c_2' from some B_j. Then

$G_1 \simeq T_1*(c_1,c_1')*(a_1,a_2)*...*(a_{n-1},a_n)$, and

$G_2 \simeq T_2*(c_2,c_2')*(b_1,b_2)*...*(b_{m-1},b_m)$ are planar d.c.s graphs containing T_1 and T_2 respectively. Now, embed G_1 and G_2 so that c_1, c_2, c_1', c_2' lie in the exterior face. Then, join c_1 and c_1' to c_2 and c_2'. Note that the resulting graph G is planar and for each vertex of degree at least 3 there is a partner u'. Hence G is d.c.s.

Case 2. m is even and n is odd.

Obviously, $d(c_1) = n+1$, which is even and

$$\{c_1, c_2, a_1, \ldots, a_n, b_1, \ldots, b_m\} \cup (\bigcup_{i=1}^{n} A_i)$$

form a tree, say T' of diameter four and $C(T') = \{c_1\}$.

Choose a vertex a_i^1 from some A_i. Now,

$$T*(a_i^1, c_i)*(a_1, c_2)*(a_2, a_3)*\ldots*(a_{n-1}, a_n)*(b_1, b_2)*\ldots(b_{m-1}, b_m)$$

is a planar d.c.s graph containing T.

Case 3. Both m and n are odd.

Here T is a spanning tree of the planar d.c.s graph,

$$T*(c_1, b_1)*(c_2, a_1)*(a_2, a_3)*\ldots*(a_{n-1}, a_n)$$
$$*(b_2, b_3)*\ldots*(b_{m-1}, b_m).$$

□

Remark 3.3. (i) In (b), if the central vertex has odd degree, the result need not be true, as seen in Fig 3.1.
(ii) There exists non solvable trees of diameter six. Also, if V_1 and V_2 are the bipartition of $V(T)$ such that $|V_1|$ is odd and each vertex of V_1 is of degree greater than 2, then T is not solvable.

We ask a problem similar to the problem discussed earlier.

PROBLEM: Find the smallest m.c.s. graph containing a given tree $T, |T| \geq 4$.

If $T = K_{1,n}; n \geq 3$, $K_{2,n}$ is such a graph and its size is 2n.

Theorem 3.4. The size of the smallest m-convex simple graph containing a tree $T \neq (K_{1,n})$ satisfies, $p-1+(m/2) \leq q \leq p+m-2$, where $|V(T)| = p$ and m is the number of pendent vertices.

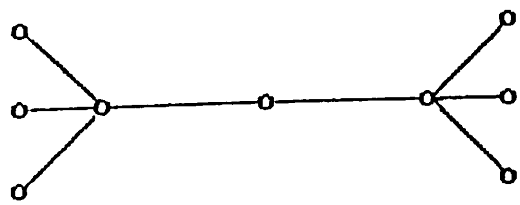
Proof. Let u_1 be a pendent vertex of T and v be the vertex adjacent to u_1. Let $u_2, u_3, \ldots, u_k$ be the other pendent vertices adjacent to v. Let $v_1, v_2, \ldots v_\ell$ be the pendent vertices other than u_is. Add edges to T such that $\{u_1,u_2,...,u_k, v_1,v_2,...,v_\ell\}$ induce a tree in which $\{u_1,u_2,...,u_k\}$ and $\{v_1,v_2,...,v_\ell\}$ is a bipartition. This is possible by taking a spanning tree of $K_{k,\ell}$. The resulting graph is triangle-free and neither a vertex nor an edge can separate G. So, by theorem 2.12, G is an m.c.s. graph and size of G is $p-1+\ell+k-1 = p+m-2$ where m is the number of pendent vertices of T. So size q of the smallest m.c.s. graph is atmost p+m-2.

Now, note that m.c.s. graphs are triangle free blocks and hence all vertices are of degree at least two. Therefore, to make T a block, the degree of each pendent vertex is to be increased by at least one. So, at least $\lceil \frac{m}{2} \rceil$ edges are to be added and hence

$q \geq p-1+\lceil \frac{m}{2} \rceil > p-1+\frac{m}{2}$. □

The following example illustrate that there are trees attaining both the bounds. Consider the tree T_1 in Fig 3.4.

T_1:



Here p=9, m=6

Fig. 3.4

The graph G in Fig 3.5 is an m.c.s. graph of size $q = 11 = p-1+\frac{m}{2}$, containing T.

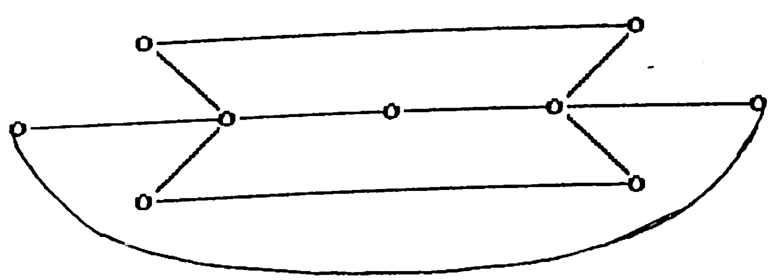


Fig. 3.5

Consider the tree T_2 of Fig 3.6. In T_2, $\{x_1, x_2\}$ is a clique such that $T_2 \setminus \{x_1, x_2\}$ is totally disconnected. So, to get an m.c.s. graph at least five edges are to be added. So, $q = 13 = p+m-2$.

T_2 :

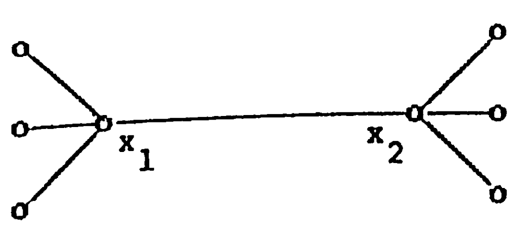


Fig. 3.6

3.2. CENTER OF DISTANCE CONVEX SIMPLE GRAPH

In this section, we determine the center of d.c.s graphs. Properties of centers of various type of graphs have been discussed by Chang [23], Chepoi [30], Nieminen [55], Prabir Das [63] and Proskurowski [64].

Theorem 3.5. If G is a planar d.c.s. graph of order at least four, then,

(1) G is self centered if diam(G) = 2.

(2) diam(G) = 2rad(G) or 2rad(G)-1, if diam(G) > 2, C(G) is isomorphic to $\bar{K}_2$ or C_4 according as diam(G) = 2rad(G) or 2rad(G)-1.)-1.

Proof: (1) Let G be a planar d.c.s. graph with diam(G) = 2. It follows from C1 of Theorem 2.1 that rad(G) > 1. So rad(G) = diam(G) and hence C(G) = V(G).

(2) Suppose diam(G) > 2.

Case I: diam(G) = 2k, k > 1

Let u,v ∈ V(G) be such that d(u,v) = 2k and $u = a_0-a_1-a_2-\ldots-a_{2k} = v$ be a shortest u-v path. Then by

C1 and theorem 2.3, we get another u-v path $u = a_0 - a_1' - \dots a_{2k-1}' - a_{2k} = v$ where a_i' and a_i are partners for $i = 1,2,\dots, 2k-1$. Note that $e(a_k) \geq rad(G) \geq k$. Let w be a vertex such that $d(a_k, w) = e(a_k)$. If $w = u$ or v then $e(a_k) = k$, which implies $e(a_k) = rad(G)$. Note that $e(a_k) = e(a_k')$.

If $w \neq u, v$ suppose that $I(v,w)$ contains a_k or a_k' (note that if $I(v,w)$ contains a_k it will contain a_k' also). Then $d(v,w) = d(v,a_k) + d(a_k,w) = k + e(a_k) \leq 2k$. This imply that $e(a_k) = k$. Similarly for $I(u,w)$. Hence in these two cases $e(a_k) = e(a_k') = rad(G)$. If neither $I(u,w)$ nor $I(v,w)$ contains these vertices, consider a shortest u-w path and shortest v-w path. Then using C1 and theorem 2.3 it can be observed that there is a subgraph homeomorphic to $K_{3,3}$. Hence $e(a_k) = e(a_k') = k = rad(G)$, that is $\{a_k, a_k'\}$ is contained in $C(G)$.

Now, we prove that these are only central vertices. If there is some other vertex, say c, in $C(G)$ then $d(c,u) \leq rad(G)$ and $d(c,v) \leq rad(G)$. But, since

$d(u,v) = 2rad(G)$, $d(c,u) = d(c,v) = rad(G)$. Thus we get a u-v path which is different from the two paths mentioned earlier. Now it can be observed that a subgraph homeomorphic to $K_{3,3}$ is contained in G.

Hence $C(G) = \{a_k, a'_k\}$.

Case II: diam (G) = 2k+1 for some k > 0.

As in the case I, if u and v are such that $d(u,v) = 2k+1$ and $u = a_0 - a_1 - \ldots - a_{2k} - a_{2k+1} = v$ and $u = a_0 - a'_1 - \ldots - a'_{2k} - a_{2k+1} = v$ are the two distinct paths then $rad(G) = k+1$ and $C(G) = \{a_k, a'_k, a_{k+1}, a'_{k+1}\}$ which will induce subgraph isomorphic to C_4. □

Remark 3.4. Planar d.c.s. graphs resembles trees in its radius-diameter relation and center-diameter relation. For a tree $T, C(T) \simeq K_1$ or K_2 according as diam(T) is 2rad(T) or 2rad(T)-1. For a planar d.c.s. graph G also, C(G) is $\bar{K}_2 \simeq D_2(K_1)$ or $C_4 = D_2(K_2)$ according as diam(G) is 2rad(G) or 2rad(G)-1. □

3.3. CONVEXITY PROPERTIES OF PRODUCT OF GRAPHS

In this section, it is proved that the property of being distance convex simple is not productive. However, m.c.s graphs behave nicely.

Theorem 3.6. Let $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ be two distance convex simple graphs. Then $G_1 x G_2$ has exactly $p_1+p_2+q_1+q_2+q_1q_2$ non trivial d-convex subsets.

Proof: Let $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ be two d.c.s graphs. Let A be a convex subset of $V(G_1 \times G_2)$.

Claim: $A = A_1 x A_2$ where $A_1 = \{u:(u,v) \in A\}$ and $A_2 = \{v:(u,v) \in A.\}$. To prove that $A_1 x A_2 \subset A$.

Let $u \in A_1$, $v \in A_2$. Then there is a $u_0 \in A_1$ and $v_0 \in A_2$ such that $(u_0,v) \in A$ and $(u,v_0) \in A$.

Let $u_0-u_1-\ldots-u_\ell-u$ be a shortest u_0-u path in G_1 and $v_0-v_1,\ldots,v_k-v$ be a shortest v_0-v path in G_2. Then $(u_0,v)-(u_1,v)-(u_2,v)\ldots(u_\ell,v),(u,v)-(u,v_k)\ldots(u,v_1)-(u,v_0)$ is a $(u_0,v)-(u,v_0)$ path. Hence $(u,v) \in A$. Therefore, $A = A_1xA_2$.

Now, even if A_i is a trivial convex set in G_i for $i = 1,2$, $A_1 x A_2$ need not be trivial. Thus the non trivial convex subsets are $\{x\} x V(G_2)$, where x is in $V(G_1)$, $V(G_1) x \{y\}$ where y is in $V(G_2)$, $\{x_1, X_2\} x V(G_2)$ where $x_1x_2 \in E(G_1)$, $V(G_1) x \{y_1,y_2\}$ where $y_1y_2 \in E(G_2)$ and $\{x_1,x_2\} x \{y_1,y_2\}$ where $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$. Number of such convex sets are p_1,p_2,q_1,q_2 and q_1q_2 respectively. Hence $G_1x G_2$ is k-convex where $k = p_1+p_2+q_1+q_2+q_1q_2$. □

Theorem 3.7 Let G_1 and G_2 be connected, triangle free graphs. $G_i \not\simeq K_1$ or K_2 for i = 1,2. Then $G_1x G_2$ is m-convex simple.

Proof: Let $G_i \simeq K_1,K_2$ be connected, triangle free graphs. Note that, if $u_1-u_2-...-u_n$ and $v_1-v_2-...-v_m$ are chordless paths in G_1 and G_2 respectively, then

$$(u_1,v_1)-(u_1,v_2)-...-(u_1,V_m)-(u_2,v_m)-...-(u_n,v_m)$$

is a chordless $(u_1,v_1)-(u_n,v_m)$ path in $G_1x G_2$.

To prove that $G_1x G_2$ is m.c.s, it is enough to prove that any (u,v) in $V(G_1 x G_2)$ is in the m-convex hull of any two nonadjacent vertices (u_1,v_1) and (u_2,v_2). Now, it can be easily seen that (u_1,v_2) and (u_2,v_1) lie on a chordless (u_1,v_1)-(u_2,v_2) path.

Assume without loss of generality that (u,v) is adjacent to (u_1,v_1).

Let u be adjacent to u_1 and $v = v_1$.

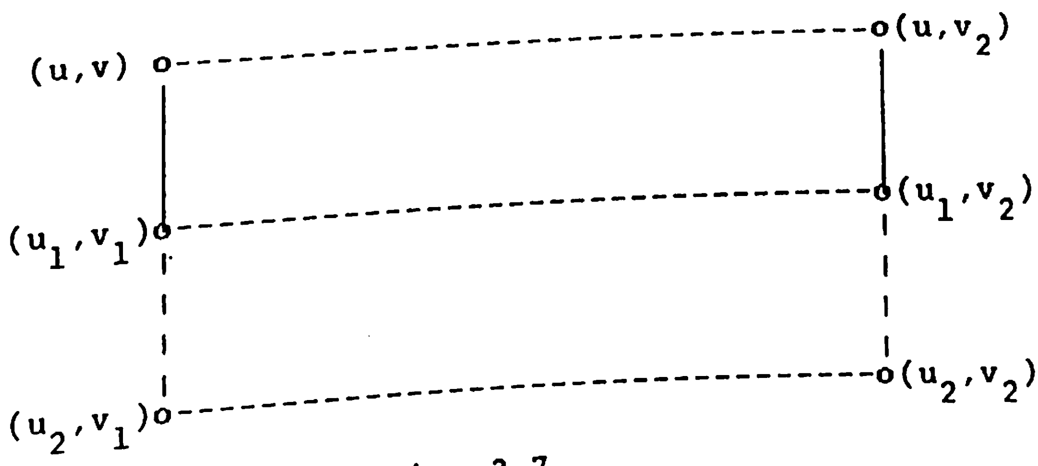


Fig. 3.7.

If u is on any chordless u_1-u_2 path say u_1-u-$a_1 \dots a_n = u_2$ then (u_1,v_1)-(u,v_1)-$(a_1,v_1) \dots (u_2,v_1) \dots (u_2,v_2)$ is a chordless (u_1,v_1)-(u_2,v_2) path containing $(u,v_1) = (u,v)$.

So assume that u is not on any chordless path connecting u_1 and u_2 (See Fig. 3.7).

Case 1. $v_1 \neq v_2$ and v_1 is not adjacent to v_2.

Then $(u_1,v_1)-(u,v_1)\ldots(u,v_2)(u_1v_2)\ldots(u_2,v_2)$ is a chordless $(u_1,v_1)-(u_2,v_2)$ path containing $(u,v_1) = (u,v)$.

Case 2. v_1 is adjacent to v_2.

Then there is vertex v_3 in G_2 different from v_1 and v_2 because $G_2 \neq K_2$. Assume v_3 to be adjacent to v_1. Then,

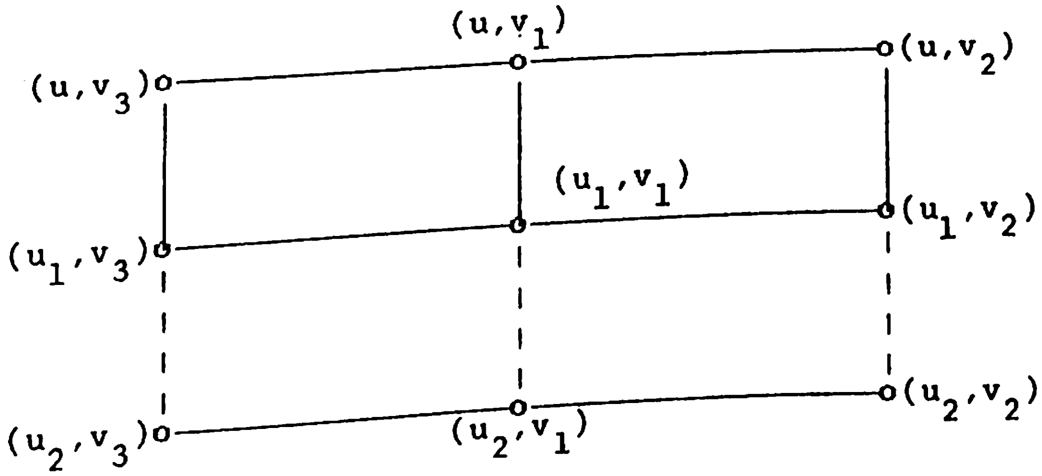


Fig 3.8

$(u_2,v_1)-(u_2,v_3)\ldots(u_1,v_3)-(u,v_3)-(u,v_1)-(u,v_2)-(u_1,v_2)$ is a chordless $(u_2,v_1)-(u_1,v_2)$ path containng (u,v) (See Fig 3.8).

That is

$(u,v) \in Co(\{(u_2,v_1),(u_1,v_2)\}) \subset Co(\{(u_1,v_1),(u_2,v_2)\})$.

If V_3 is adjacent to v_2, then

$(u_1,v_1)-(u,v_1)-(u,v_2)-(u,v_3)-(u_1,v_3)\ldots(u_2,v_3)-(u_2,v_2)$

is a chordless $(u_1,v_1)-(u_2,v_2)$ path containing

$(u,v_1) = (u,v)$.

Case 3. $v_1 = v_2$. Then $(u_2,v_2) = (u_2,v_1)$ and u_1 is not adjacent to u_2. Since $G_2 \ncong K_1,K_2$ there are two vertices v_3 and v_4 in G such that $<\{v_1,v_3,v_4\}>$ is connected.

Let v_3 be adjacent to v_1 and v_4. Then

$$(u_1,v_1)-(u,v_1)-(u,v_3)-(u,v_4)-(u_1,v_4)\ldots(u_2,v_4)$$
$$-(u_2,v_3)-(u_2,v_1)$$

is a chordless $(u_1,v_1)-(u_2,v_1)$ path containing

$(u,v_1) = (u,v)$.

Now, let v_1 be adjacent to v_3 and v_4. (See Figure 3.9.)

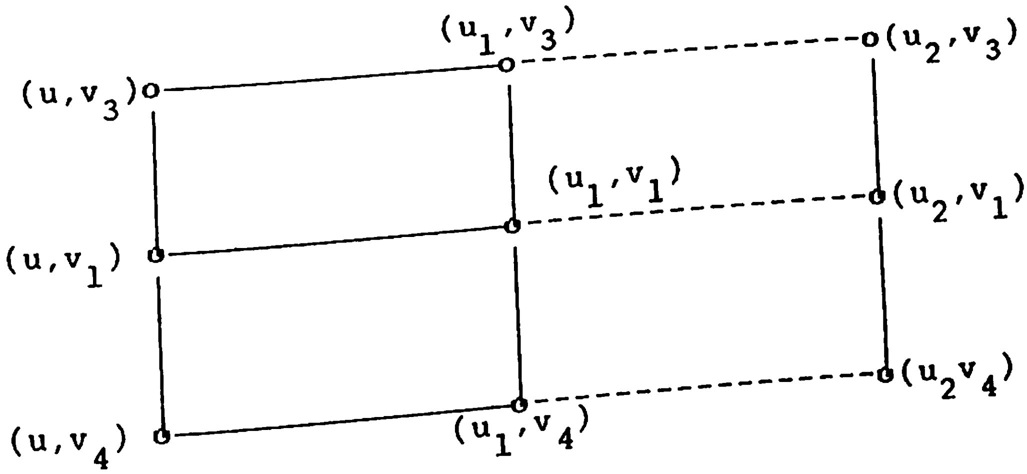


Fig 3.9

From Fig.3.9. it is clear that (u_1,v_3) and (u_1,v_4) lies on some chordless path connecting (u_1,v_1) and (u_2,v_1) because u_1 is not adjacent to u_2. Then

$$(u_1,v_3)-(u,v_3)-(u,v)-(u,v_4)-(u_1,v_4)$$

is a chordless $(u_1,v_3)-(u_1,v_4)$ path.

Hence $(u,v) \in Co(\{(u_1,v_3),(u_1,v_4)\}) \subset Co(\{(u_1,v_1),(u_2,v_1)\})$.

Hence, in any case $(u,v) \in Co(\{(u_1,v_1)(u_2,v_2)\})$ and so $G_1 x\ G_2$ is m.c.s. □

Theorem 3.8. Let G_i for $i = 1,2$ be connected triangle free graphs, where G_1 is 2-connected, $G_2 \neq K_1$, then $G_1 x\ G_2$ is m.c.s

As in the proof of theorem 2.16, let (u_1,v_1) and (u_2,v_2) be two non adjacent vertices of $G_1 \times G_2$.

Let $(u,v) \in G_1 \times G_2$. Assume (u,v) to be adjacent to (u_1,v_1).

Let $u = v_1$ and u_1 is adjacent to u.

If v_1 is not adjacent to v_2, then as in the above theorem $(u,v) \in Co(\{(u_1,v_1)(u_2,v_2)\})$.

Case I. v_1 adjacent to v_2.

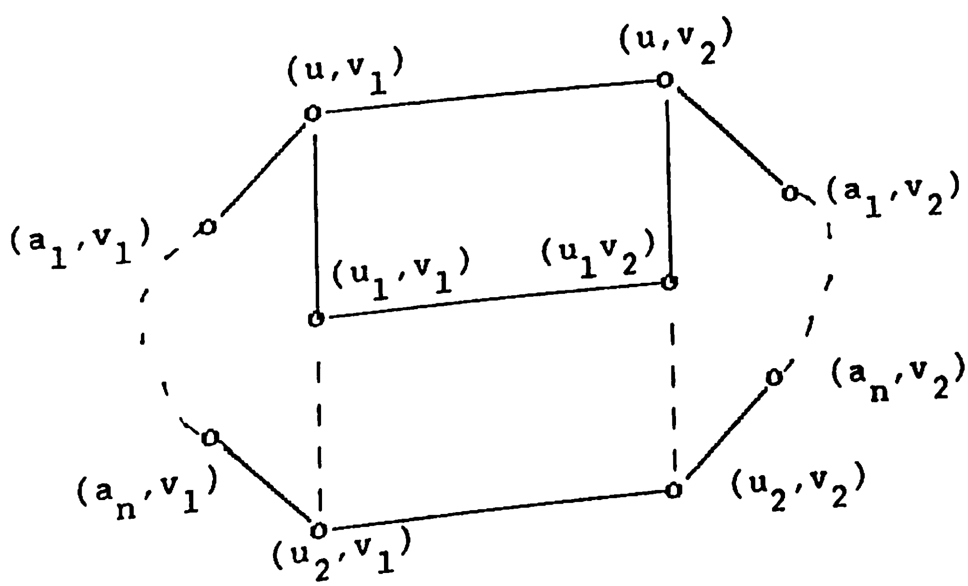


Fig. 3.10.

Since G_1 is 2-connected, there is a path connecting u and u_2 distinct from the path u-u_1-...-u_2.

Let it be $u-a_1-a_2 \ldots a_n = u_2$. Then

$$(u_1,v_1)-(u,v_1)-(u,v_2)-(a_1,v_2)\ldots(a_n,v_2)-(u_2,v_2)$$

is a chordless path.

Case II $v_1 = v_2$. Then $(u_2,v_2) = (u_2,v_1)$ and u_1 is not adjacent to u_2. Since G_2 K, there is a vertex v_2 adjacent to v_1. Then

$(u_1,v_1)-(u,v_1)-(u,v_2)-(a_1,v_2)\ldots(a_n,v_2)-(u,v_2)-(u_2,v_1)$ is a chordless $(u_1,v_1)-(u_2,v_1)$ path. □

Now let v be adjacent to v_1 and $u = u_1$. Then if $v = v_2$, then, $(u_1,v_1)-(u_1,v_2)-(u_2,v_2)$ is a chordless path containing $(u_1,v_2) = (u\ ,v)$.

If $v \neq v_2$, then v, v_1 and v_2 are distinct vertices of G_2 and hence $G_2 \neq K_2$ or K_1. Then the theorem holds as in Theorem 3.7. Now if $v_1 = v_2$ and v is adjacent to v_1, then u_1 is not adjacent to u_2. (See Fig. 3.11).

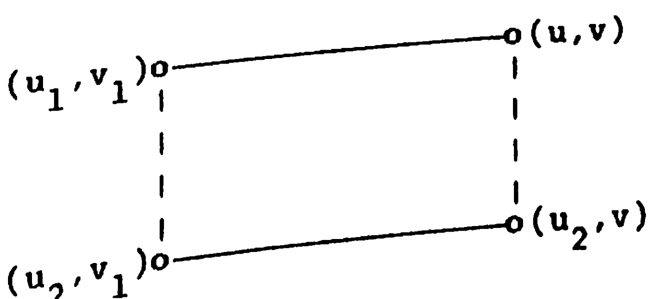


Fig. 3.11.

In this case $(u_1,v_1)-(u_1,v)\ldots(u_2,v)-(u_2,v_1)$ is a $(u_1,v_1)-(u_2,v_1)$ path containing $(u_1,v) = (u,v)$. □

Remark 3.5. The condition that G_1 is 2-connected is necessary. For, taking G_2 be K_2 and G_1 to be a graph having a cut point c, then $G_1 \times G_2$, the copy of K_2 corresponding to c will be a clique separator for $G_1 \times G_2$ and hence $G_1 \times G_2$ will not be m.c.s.

Theorem 3.9. If G_1 is an m.c.s graph and G_2 is any connected triangle free graph, then $G_1 \times G_2$ is m.c.s

Proof: If $G_2 \simeq K_1$, then $G_1 \times G_2 \simeq G_1$ and hence is m.c.s.

If $G_2 \simeq K_1$, then using theorem 2.17, $G_1 \times G_2$ is m.c.s. □