CHAPTER IV

IV

CONVEXITY FOR THE EDGE SET OF A GRAPH

In this chapter we introduce a notion of convexity for the edge set of a connected graph. This definition is motivated by the concept of edge lattice of a graph discussed in [4]. Though there is a vast literature concerning different aspects of convexity for the vertex set of a graph, little work is done on similar lines for the edge set.

We first observe that this convexity on E(G) in addition satisfies the exchange law and hence is a matroid (Definition 1,15). Also, its arity is not in general two and hence the convexity is not induced by an interval. It is known that the Caratheodory number of a convex structure is an upper bound for its arity.

In this chapter, we have evaluated the convex invariants of this convex structure. The Pasch Peano properties (Definition 1.20) are also discussed and also a forbidden subgraph characterization. Some results of this chapter are in [61].

4.1 CYCLIC CONVEXITY

Definition 4.1 Let $G = (V,E)$ be a graph with $E \neq \phi$. $S \subseteq E$ is cyclically convex if it contains all edges comprising a cycle whenever it contains all but one edge of this cycle.

Equivalently if S is convex and if $a_1a_2, a_2a_3, \ldots, a_{n-1}a_n \in S$ and $a_na_1 \in E$ then a_na_1 also will be in S where a_ia_{i+1} is an edge of G for $i = 1,2,\ldots,n-1$.

If $\mathscr{C}$ denotes the collection of all such convex subsets of E, then $(G,\mathscr{C})$ is convexity space. For convenience, the cyclic convexity on E will also be referred to as convexity.

Example : (a) For a tree T, every subset of $E(T)$ is trivially convex.

(b) In the graph G of Fig 4.1,

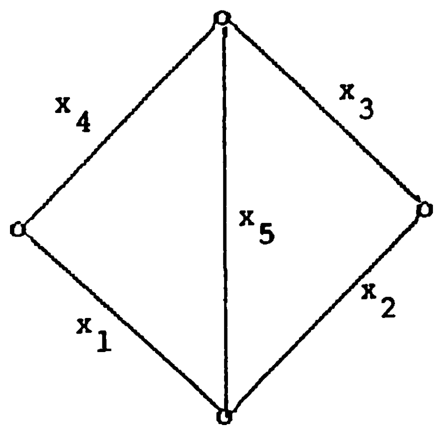

Fig 4.1

$\{x_1\}, \{x_1, x_2\}$ are convex but $\{x_1, x_4\}$ is not.

Now, we shall consider a generalization of the notion of geodetic iteration number (Definition 1.11) of an interval convexity space to a convexity space of arity greater than two.

Definition 4.2. Let X be a convexity space of arity n (n>2) and $S \subset X$. The closure of S, denoted by (S) is defined as $(S) = \bigcup\{Co(F) : F \subset X, |F| \leq n\}$. S^m is recursively defined as, $S^1 = (S)$, $S^m = (S^{m-1})$. The smallest positive integer m such that $S^m = S^{m+1}$ is called the iteration number of S. The iteration number of X is defined to be max{iteration number of S : $S \subseteq X$} if it exists.

Lemma 4.1. For the convexity space $(G,\mathcal{C})$, the iteration number is equal to 1.

Proof: We shall prove that for $S \subset E$, $S^2 = S^1$. It is obvious that $S^1 \subset S^2$. Let $e \in S^2 = (S^1)$. Then, there is a sequence of edges say $e_1 = a_1a_2$, $e_2 = a_2a_3, \ldots, e_{n-1} = a_{n-1}\,a_n$ in S^1 such that $\{e_1, e_2, \ldots, e_{n-1}, e\}$ forms a cycle in G. Then, for each $i = 1,2,\ldots,n-1$, we get,

$$S_i = \{e_{i1} = a_i a_i^1,\ e_{i2} = a_i^1\ a_i^2, \ldots, e_{ik_i} = a_i^{k_i}\ a_{i+1}\} \subset S$$

such that $\{e_i, e_{i1}, \ldots, e_{ik_i}\}$ comprise a cycle in G. Now, observe that $\bigcup S_i$ is a sequence in S_i which contains a subsequence $e^1, e^2, \ldots, e^m$ forming a path joining a_1 and a_n. Hence, $\{e, e^i, \ldots, e^m\}$ forms a cycle in G and so $e \in S^1$. Thus, $S^2 = S^1$. □

Theorem 4.2 The arity of $(G,\mathcal{C})$ is 1 if G is a tree and is one less than the size of the largest minimal cycle in G, otherwise.

Proof: If G is a tree, then the lemma is trivially true. So assume G to be a graph having a cycle. Let k be the size of the largest minimal cycle. Let $S \subset E(G)$ is such that $Co(F) \subset S$ for $F \subset S, |F| \leq k-1$. Let $e \in Co(S)$. Then by lemma 4.1, there is a sequence $\{e_1, e_2, \ldots, e_t\}$ of edges in S such that $\{e, e_1, \ldots, e_t\}$ comprise a cycle. If $\{e_1, e_2, \ldots, e_t\}$ comprise a minimal path, then, $\{e, e_1, \ldots, e_t\}$ comprise a minimal cycle and hence $t \leq k-1$. Hence, $e \in Co(\{e_1, e_2, \ldots, e_t\}) \subset S$. If $e_1, \ldots, e_t$ comprise a path having a chord, assume that, $Co(F) \subset S$ for $|F| < t$. Let e_0 be a chord of this path such that there is a sequence $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$, $i_1, \ldots, j_i \in \{1, 2, \ldots, t\}$ and $\{e_0, e_{i_1}, \ldots, e_{i_j}\}$ comprise a minimal cycle.

Then $e_j \leq k-1$ and $e_0 \in Co(e_{i_1}, \ldots, e_{i_j}) \subset S$.

Now, $\{e, e_0, e_1, \ldots, e_t\} \setminus \{e_{i_1}, \ldots, e_{i_j}\}$ comprises a cycle of length less than $t+1$ and $e \in Co(\{e_0, e_1, \ldots, e_t\}) \subset S$ by induction hypothesis. Hence, arity of $(G, \mathscr{C}) \leq k-1$.

Now, if C_k is some largest minimal cycle in G, then let $S = E(C_k) \setminus x$, where x is an edge in the cycle. Then S is with the property that $Co(F) \subset S$ for each subset of cardinality at most k-2, but S is not convex. Hence, the arity of $(G,\mathscr{C})$ is one less than the length of some largest minimal cycle in G. Hence arity $A(G,\mathscr{C}) = k-1$. □

We shall now consider the concept of rank of a matroid. For a convex structure X, a nonempty subset $F \subseteq X$ is convexly independent provided $x \notin Co(F \setminus \{x\})$ for each $x \in F$. Further, if X is a matroid (Definition 1.15) there exists a maximal independent subset of X and such a set is called a basis of the matroid. The cardinality of the basis is called the rank.

Theorem 4.3[12]. In a matroid X the hull of a basis equals X and all bases of X have the same cardinality.

Now we prove the following,

Theorem 4.4. If G is a connected graph, $(G,\mathscr{C})$ is a matroid of rank p-1, where $p = |V(G)|$.

Proof: $(G,\mathscr{C})$ is a matroid follows from the fact that if

$\{p,q,x_1,\ldots,x_n\}$ comprise a cycle, $p \in Co(\{q,x_1,\ldots,x_n\})$ and $q \in Co(\{p,x_1,\ldots,x_n\})$.

Now, we have to prove that rank (G) = p-1. Let T be a spanning tree of G and let F = E(T). Then each pair of vertices in G is connected by a path in T. Now if $e \in E(G)$ such that $e \notin E(T)$ then there is a sequence $e_1,\ldots,e_n$ of edges in F connecting the end vertices v_1 and v_2. That is $\{e_1,e_2,\ldots,e_n\}$ comprise a cycle. Hence E(G) = Co(F). Hence, rank (G) $\leq$ p-1.

Now, let $F \subset E$, be such that $|F| < p-1$. Then there are two vertices v_1 and v_2 in G such that it is not connected by a path comprised by edges in F. If $e_1,\ldots,e_k$ are those edges in G which comprise a path joining v_1 and v_2 and if $e_1,\ldots,e_k \subset Co(F)$ then by lemma 4.1 there is a sequence of edges in F which comprise a v_1-v_2 path which is a contradiction □

Corollary: If G is disconnected, then (G,$\mathcal{C}$) is a matroid of rank p-k where k is the number of components of G. □

4.2 CONVEX INVARIANTS

The convex invariants (Definition 1.18) of $(G,\mathcal{C})$ will be denoted by $h(G)$, $c(G)$, $r(G)$ and $e(G)$ respectively.

Theorem 4.5. If G is a connected graph of order p, the Helly number of $(G,\mathcal{C})$ is $p-1$.

Proof: Let T be a spanning tree of G and $F = E(T)$. We shall prove that F is H-independent.

Let $e^1 \in Co(F\setminus\{e\})$ for every e in F. Then by the lemma 4.1, there is a sequence of edges, $e_1 = a_1a_2$, $e_2 = a_2a_3$, $\ldots$, $e_{n-1} = a_{n-1}a_n$ in $F\setminus\{e\}$ such that $e^1 = a_1a_n$, for some e in F. Then $e_1 \in F$ and $e^1 \in Co(F\setminus\{e_1\})$ also. Again using lemma 4.1, we get another sequence $e_{1,1} = a_1a_{1,2}$, $e_{1,2} = a_{1,2}a_{1,3}$, $\ldots$, $e_{1,k} = a_{1,k}a_n$. This is contradiction, since $e_1, e_1, \ldots, e_{n-1}$ and $e_{1,1}, \ldots, e_{1,k}$ will then comprise two distinct paths joining a_1 and a_n of T. Hence, $\bigcap \{Co(F-\{a\})/a \in S\}$ is empty and so $h(G) \geq p-1$.

Now, we prove that any subset F of cardinality at least p is H-dependent. In this case F contains a subset,

$C = \{e_1, e_2, \ldots, e_k\}$, comprising a cycle in G and $e_i \in Co(F \setminus \{e\})$ for each e in F and $i = 1,2,\ldots,k$. Hence, $\bigcap \{Co(F \setminus \{e\}) / e \in F\}$ is not empty. Therefore, F is H-dependent and so $h(G) < p$. Thus, $h(G) = p-1$. □

Theorem 4.6. The Caratheodory number of $(G, \mathscr{C})$ is given by

$$c(G) = \begin{cases} 1 & \text{if } G \text{ is a tree} \\ Circ(G)-1, & \text{otherwise, where } Circ(G) \text{ is the circumference of } G. \end{cases}$$

Proof: If G is a tree, then every subset of E is convex. Hence, for each $F \subset E$ with cardinality at least two, we have,

$Co(F) = F \subset \bigcup_{e \in F} (F - \{e\}) = \bigcup_{e \in F} Co(F \setminus \{e\})$. Hence, $c(G) = 1$.

Now, let C be a longest cycle in G of length k and $S = E(C) = \{a_1a_2, a_2a_3, \ldots, a_{k-1}a_k, a_ka_1\}$. Then $a_ia_{i+1} \in Co(S \setminus \{a_ia_{i+1}\})$ for each $i = 1,2,\ldots,k$.

Let $S_i = (S - \{a_ia_{i+1}\})$.

Claim: $a_ia_{i+1} \notin Co(S_i \setminus \{e_i\})$ for $e_i \in S_i$.

If $a_ia_{i+1} \in Co(S_i \setminus \{e_i\})$, by the lemma 4.1 we get a sequence $e_1, e_2, \ldots, e_{k_i}$ of edges in $S_i - \{e_i\}$ such that $\{a_ia_{i+1}, e_1, \ldots, e_{K_i}\}$ comprise a cycle in G. This is not possible because $S-\{e_i\}$ consists of the edges of a path only. Hence, $Co(S_i) \subset \bigcup Co(S_i \setminus \{e_i\} : e_i \in S_i\}$ and so $c(G) \geq k-1$.

Now, let S be a subset of E of cardinality at least k. Let $e \in Co(S)$. If $e \in S$, $e \in S-\{e^1\} \subset Co(S \setminus \{e^1\})$, for some $e^1 \neq e$ in S. If $e \notin S$, there is a sequence $e_{1,1},\ldots,e_{1,\ell}$ in S such that $\{e,e_{1,1},\ldots,e_{1,\ell}\}$ comprise a cycle in G. Also, $S \neq \{e_{1,1},\ldots,e_{1,\ell}\}$ because of the maximality of C. Let $e^1 \in S - \{e_{1,1},\ldots,e_{1,\ell}\}$. Then $e \in Co(S \setminus \{e^1\})$ and so $Co(S) \subset \bigcup Co(S \setminus \{e^1\} / e \in S)$ and $c(G) \leq k-1$. Hence, $c(G) = k-1$. □

Theorem 4.7. If G is a connected graph of order p, the Radon number of $(G,\mathcal{C})$ is $p-1$.

Proof. Let T be a spanning tree and let $F=E(T)$. Then if F can be partitioned into F_1 and F_2 such that $Co(F_1) \cap Co(F_2) \neq \phi$ and if $e \in Co(F_1) \cap Co(F_2)$, then there is a sequence of edges $e_{11},\ldots,e_{1\ell}$ in F_1 and $e_{21},\ldots,e_{2m}$ in F_2 such that $e,e_{11},\ldots,e_{1\ell}$ and $e,e_{21},\ldots,e_{2,m}$ comprise cycles. Then $e_{11},\ldots,e_{1\ell}$ and $e_{21},\ldots,e_{2m}$ are paths connecting the end vertices of e and hence $\{e_{11},\ldots,e_{1\ell}, e_{21},\ldots,e_{2m}\}$ contains a sequence comprising a cycle, which is not possible. So F cannot have a Radon partition. Hence, $r(G) \geq p-1$.

Now, let $F \subset E(G)$ be of cardinality greater than $p-1$. Then it contains a subsequence $\{e_1,\ldots,e_s\}$ comprising a cycle C. Then for $e \neq e_i$, $e_i \in Co(F \setminus \{e\})$ for $i = 1,\ldots,s$. Also $e_i \in Co(E(C) \setminus \{e_i\}) \subset Co(F \setminus \{e_i\})$. Now, let $F = F_1 \cup F_2$ be such that $E(C) \setminus \{e_i\} \subset F_1$ and $\{e_i\} \subset F_2$. Then $e_i \in Co(F_1) \cap Co(F_2)$. Hence, $r(G) \leq p-1$. Therefore, Radon number $r(G) = p-1$. □

Theorem 4.8. For a connected graph G, the exchange number is given by

$$e(G) = 2 \text{ if } G \text{ is a tree or a cycle}$$
$$= \max\{Circ(G-v) : v \in V(G)\}, \text{ otherwise.}$$

Proof:

Case I: Let G be a tree. In this case, every subset F of $E(G)$ is convex. If $|F| \leq 2$ then let $F = \{e_1,e_2\}$. Then $F \setminus \{e_1\} \not\subset F \setminus \{e_2\}$, hence F is E-independent. If $|F| \geq 3$, let $F = \{e_1,\ldots,e_n,p\}$, $n \geq 2$. Then,

$$\begin{aligned} Co(F \setminus \{p\}) &= F \setminus \{p\} = \{e_1, \dots, e_n\} \\ &= \{e_1,\dots,e_{n-1}\} \cup \{e_1,\dots,e_{n-2},e_n\} \cup \\ &\quad \dots \cup \{e_1,e_3,\dots,e_n\} \cup \{e_2,\dots,e_n\} \\ &\subset \cup \{ F \setminus \{e_i\}) : i=1,..,n\}. \end{aligned}$$

Hence, $Co(F\setminus\{p\} \subset \cup \{CoF \setminus \{e\}) : e \neq p,\ e \in F\}$.

Case II: Let G be a cycle. Then either F=E or F has no subsequence comprising a cycle.

If F = E, $Co(F\setminus \{e\}) = F$ for each e in F. If $F \neq E$, since F contains no sequence comprising a cycle, each proper subset of F is convex and so proof is as in the case of a tree. Hence for both the cases, the exchange number is 2.

Case III: G is a graph having a cycle \`C' and a vertex v not in \`C'.

Assume without loss of generality that \`C' is the longest cycle with this property and let v be a vertex not in C. Let $C = a_1-a_2-a_3-\dots a_n-a_1$, $a_i \in V$ for $i = 1,2,\dots,n$. Let u be a vertex adjacent to v and let $S = \{a_1a_2, a_2a_3, \dots, a_{n-1}a_n, uv\}$. Then it is clear that $a_na_1 \in Co(S)$.

Claim: $a_n a_1 \notin Co(S \setminus \{a_i a_{i+1}\})$ for any $i=1,2,\ldots,n$. If not, $(S \setminus \{a_i a_{i+1}\}) \cup \{a_n a_1\}$ contains a sequence comprising a cycle, which is not possible. Hence $a_n a_1 \notin Co(S \setminus \{a_i a_{i+1}\})$ for any i. Hence S is E-independent and the exchange number is at least the cardinality of S, which is equal to n.

Now, let S be a subset of cardinality atleast $n+1$, say $S = \{e_1, \ldots, e_m\}$, $m \geq n+1$.

Let $e \in Co(S \setminus \{e_i\})$ for some i.

To prove that $e \in Co(S \setminus \{e_j\})$ for some $j \neq i$.

Since $e \in Co(S \setminus \{e_i\})$ by lemma 4.1, we get a sequence $e'_1, \ldots, e'_k$ in $S \setminus \{e_i\}$ Such that $e, e'_1, \ldots, e'_k$, comprise a cycle.

If $S \setminus \{e_i\} = \{e'_1, \ldots, e'_k\}$, then $S \cup \{e\} \setminus \{e_i\}$ comprise a cycle of length $m \geq n+1$ and it contradicts the maximality of C.

So, there is a subsequence of $S \setminus \{e_i\}$ say $f_1, f_2, \ldots, f_\ell$ such that $F = \{e, e_i, f_1, f_2, \ldots, f_\ell\}$ comprise a cycle. Let $f \in S \setminus F$, then $e \in Co(S \setminus \{f\})$. Hence S is E-dependent and so $e(G) < n+1$, Thus $e(G)=n$. □

These theorems are illustrated in Fig 4.2.

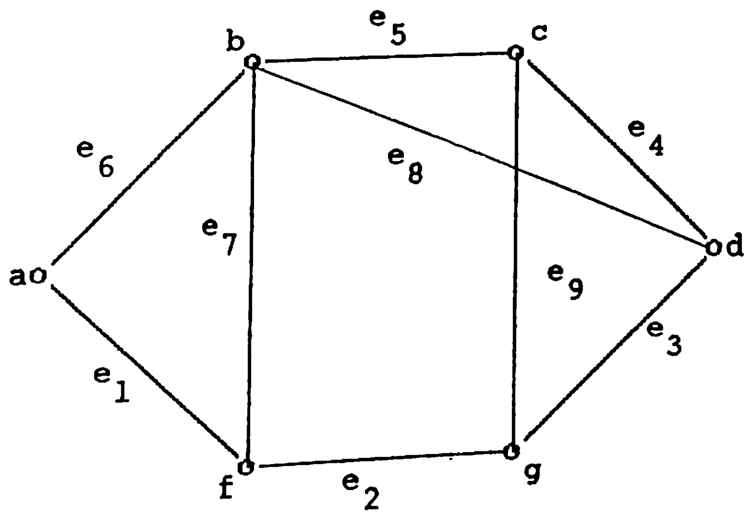

Fig. 4.2

In figure 4.2, Circ(G) = 6, max { Circ(G-v): v $\in$ G}=5.

Let F = $\{e_1, e_2, e_3, e_4, e_5\}$

Then Co(F) = E(G),

$Co(F \setminus \{e_1\}) = \{e_2, e_3, e_4, e_5, e_7, e_8, e_9\}$,

$Co(F \setminus \{e_2\}) = \{e_1, e_3, e_4, e_5, e_8, e_9\}$,

$Co(F \setminus \{e_3\}) = \{e_1, e_2, e_4, e_5, e_8\}$,

$Co(F \setminus \{e_4\}) = \{e_1, e_2, e_3, e_5\}$ and

$Co(F \setminus \{e_5\}) = \{e_1, e_2, e_3, e_4, e_9\}$.

Also $\bigcap \{ co\ F \setminus \{e_i\} \mid i=1,\dots,5\}$ is empty.

So, F is an H-independent set. Actually, it is a maximal H-independent set and hence,

$h((G,\gamma)) = 5 = 6-1 = p-1$.

F is R-independent, because for any partition F_1 and F_2 of F, $Co(F_1) \cap Co(F_2) = \phi$. Hence F is an R-independent set and it is maximal. So $r((G,\mathscr{C})) = 5 = p-1$.

F is C-independent because $e_6 \in Co(F)$ and $e_6 \notin Co(F \setminus \{e_i\})$ for any i=1,2,3,4,5. Also F is maximal. Hence $C((G,\mathscr{C}))=5$.

F is E-independent because $e_7 \in Co(F \setminus \{e_1\})$ and $e_7 \notin Co(F \setminus \{e_i\})$ for i=2,3,4,5. Here also F is maximal. Hence $C((G,\mathscr{C})) = 5$.

Note 4.1. (a) In this example, we have h = c = r = e = 5.

(b) If the graph G is Hamiltonian, then

h = c = r.

4.3 PASCH-PEANO PROPERTIES

In this section we shall consider the Pasch Peano properties (Definition 1.20). It is possible to express the Pasch Peano properties of a general convexity space by replacing the interval operator by the convex hull operator.

Here we discuss the Pasch Peano properties of $(G,\mathscr{C})$.

Definition 4.3. A convexity space X has Pasch property if, for $a,b,t,a^1,b^1 \in X$ such that $a^1 \in Co(\{a,t\})$, $b^1 \in Co(\{b,t\})$, then $Co(\{a,b^1\}) \cap Co(\{a^1,b\}) \neq \phi$ and X has Peano property if for a,b,d,u,v in X such that $u \in Co(\{a,b\})$, $v \in Co(\{d,u\})$, there is a `w' in $Co(\{b,d\})$ such that $v \in Co(\{a,w\})$.

We shall denote the edges of G by a,b,d,f and g.

Theorem 4.9. The convex structure $(G,\mathscr{C})$ is a Pasch space if and only if $K_4 - x$ is not an induced graph of G.

Proof: If $K_4 - X$ is a graph, let u,v,w,t be such that uv = a, vt = f, uw = d, vw = g and wt = b are in E and ut $\notin$ E (See Fig 4.3).

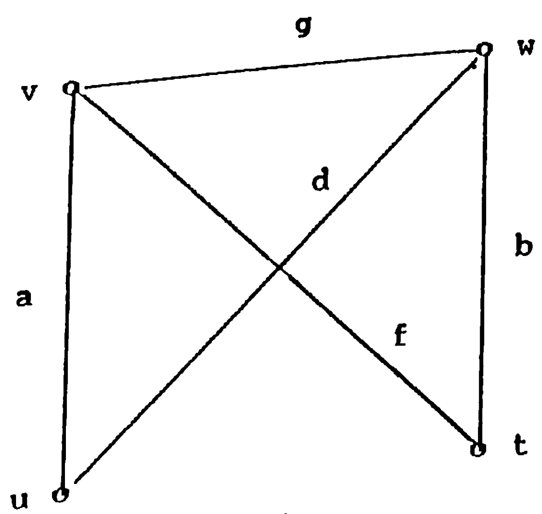

Fig 4.3

Then $d \in Co(\{a,g\})$, $f \in Co(\{b,g\})$ and $Co(\{a,f\}) \cap Co(\{b,d\}) = \{a,f\} \cap \{b,d\} = \phi$.

Now assume that K_4 -x is not a subgraph. Let $a,b,g,d,f \in E$ be such that $d \in \in Co(\{a,g\})$, $f \in Co(\{a,g\})$.

If $d \neq a, g$: $f \neq b, g$ then a,b,d,f and g will be as shown in the figure 2. Since K_4 - x is not an induced subgraph, $ut \in E$ and $ut \in Co(\{a,f\}) \cap Co(\{b,d\})$. If d=a (or if f=b), clearly $Co(\{b,d\}) \cap Co(\{a,f\}) \neq \phi$. Now if d=g, then $f \in Co(\{b,g\}) = Co(\{b,d\})$ and hence $Co(\{a,f\}) \cap Co(\{b,d\}) \neq \phi$. Hence the theorem. $(G,\mathcal{C})$ is Pasch if and only if K_4 - x is not an induced subgraph of G. □

Theorem 4.10. The convex structure $(G,\mathcal{C})$ is a Peano space if and only if G does not contain K_4 - x as a subgraph.

Proof: Let G contain K_4 - x as a subgraph. Then G contains a subgraph isomorphic to the graph in figure 4.4 .

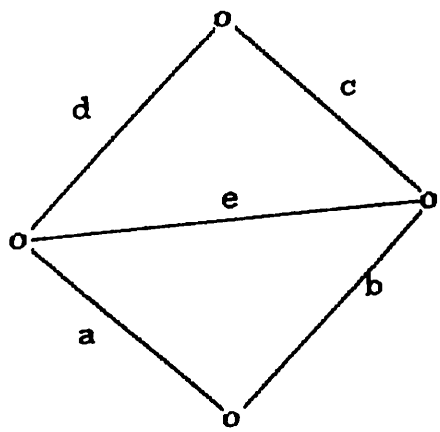

Fig. 4.4

In G, a,b,e,d,f are such that e $\in$ Co({a,b}), f $\in$ Co({e,d}). But it is not possible to find a `g' in Co({b,d}) ={b,d} such that f $\in$ Co({a,g}).

Now, let G be graph which contain no subgraph isomorphic to K_4 -x. Let a,b,d,e,f be as in the Peano condition.

Let e $\in$ Co({a,b}). If e=a or b, then the proof is trivial. So assume e $\neq$ a or b. If Co({e,d}) = {e,d}, then f = e or d and belongs to Co({a,b}) or Co({a,d}). If Co({e,d}) $\neq$ {e,d}, there is an f $\neq$ e,d in Co({e,d}). Then f is adjacent to e and d and so {a,b,d,e,f} comprise a K_4 - x which is not possible. Hence the theorem.

Note 4.2. It can be easily observed that for matroids Peano property implies the Pasch property. In particular, (G,γ) is a Peano space implies that it is a Pasch space. The converse is not true. (K_4,γ) is a pasch space which is not a Peano space, by theorem 4.9 and 4.10. □