SOME PROPERTIES OF H-CONVEXITY ON R^n.

In this chapter, we consider some problems posed by Van de Vel [12] on the H-convexity of R^n. This convexity on vectorspaces generated by linear functionals has been studied by Boltyanskii [19] and Bourguin [20] and has some interesting properties. In general, a symmetrically generated H-convexity need not be JHC or S_4. In the process of answering a Problem of Van de Vel ([12] and also on a recent private communication), as to whether each symmetric H-convexity is of arity two, we obtain a sufficient condition for a symmetrically generated H-convexity to be of arity two and give an example to illustrate that the arity could be infinite. A necessary and sufficient condition for the symmetrically generated H-convexity to be S_4, and an example of a PP space which is neither JHC nor S_4 and hence not of arity two are also obtained.

5.1 H-CONVEXITY

Let V be a vectorspace over a totally ordered

field K and let $\mathscr{F}$ be a collection of linear functionals from V $\rightarrow$ K. Then the family $\mathscr{S}$ = {$f^{-1}(-\infty,a]$: a $\in$ k, f $\in$ $\mathscr{F}$} generates a convexity $\mathscr{C}$ on V, coarser than the standard one. It is called an H-convexity. If -f $\in$ $\mathscr{F}$ whenever f $\in$ $\mathscr{F}$ then $\mathscr{C}$ is called a symmetric H-convexity. We usually omit one of f, -f and say that $\mathscr{F}$ symmetrically generate the convexity $\mathscr{C}$. The usual convexity in R^n is an H-convexity generated by the collection of all linear functionals from $R^n \rightarrow R$.
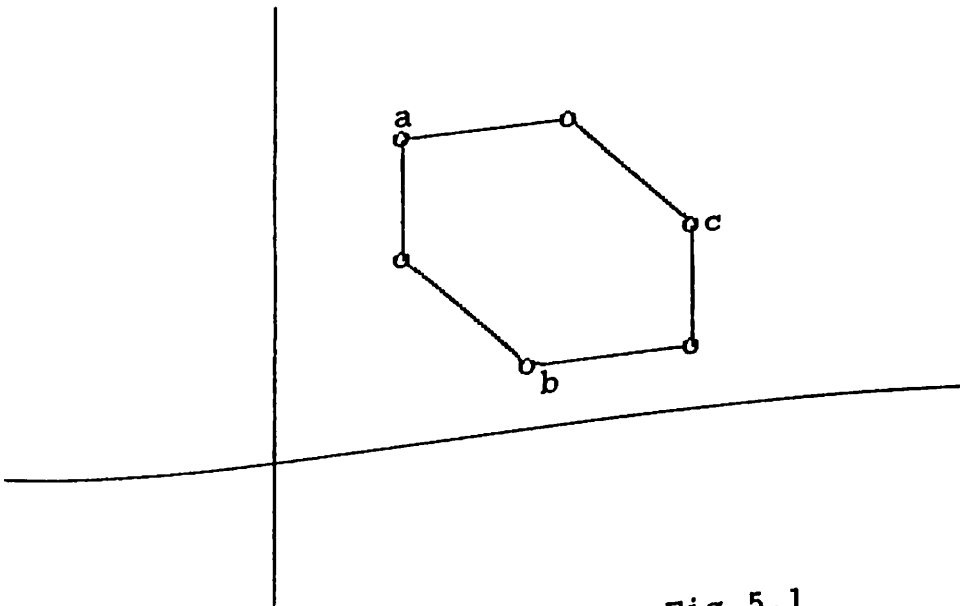

Fig.5.1

Figure 5.1 gives a typical polytope of R^2 generated by the

co-ordinate projections and their sum in which {a,b,c} is a spanning set. Observe that the standard convex hull of {a,b,c} is the triangle with vertices a,b, and c and is contained in this polytope.

Let X and Y be two convexity spaces. A function $f: X \to Y$ is a convexity preserving function (CP function) if for each convex set $C \subset Y$, $f^{-1}(C)$ is convex. A function f is convex to convex (C C function) if for each convex set $C \subset X$, $f(C)$ is convex.

If X is R^n with usual convexity and Y is R^n with an H-convexity then the identity mapping from $X \to Y$ is a CP function.

A symmetrically generated H-convexity need not be JHC or S_4.

Example [12].

Let $\mathcal{E}$ be the H-convexity symmetrically generated by the co-ordinate projections f_i and their sum, defined on R^3. $\mathcal{F} = [f_1, f_2, f_3, f_4 = f_1 + f_2 + f_3]$.

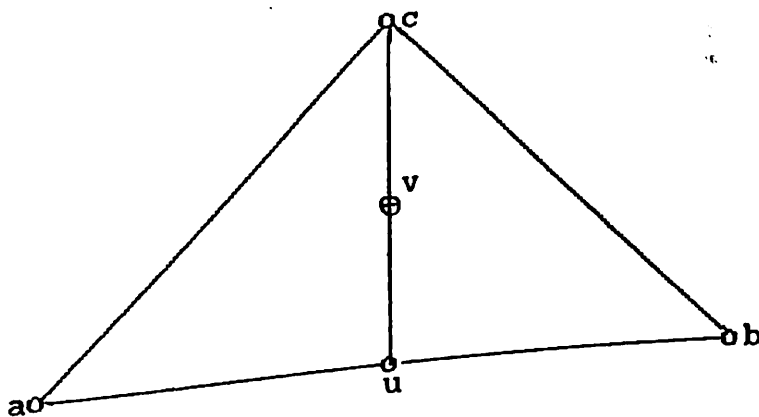Let a = (o,3/4,1/4), b = (1/2,1/4,0), c =(0,0,1/2)

u = (1/2,1/4,1/4), v = (1/2,0,1/2).

Fig. 5.2.

Then there does not exist a $v^1 \in Co\{a,c\}$ such that $v \in Co(\{b,v^1\})$. If such a v^1 exists, then

$0 \leq f_1(v^1) \leq 0$, hence $f_1(v^1) = 0$

$f_2(v^1) \leq 0 \leq 1/4$ hence $f_2(v^1) = 0$

$1/4 \leq f_3(v^1) \leq 1/2$, hence $f_3(v^1) \leq 1/2$

and therefore $f_4(v^1) \leq 1/2$.

But $f_4(v) = 1$, $f_4(b) = 3/4$. So there can not exists 'v^1' in Co ($\{a,c\}$) such that $v \in Co(\{b,v^1\})$.

That is, $\mathscr{C}$ does not satisfy the Peano property and hence is not JHC.

Example 5.2 Let $C_1 = \{(x,y,z): x \leq 0, y \leq 0\}$ and

$$C_2 = \{(x,y,z): z \leq -1, x+y+z \geq 0\}.$$

Then C_1 and C_2 are disjoint convex sets which cannot be separated by half spaces. That is, the H-convexity is not in general S_4. For another example, see [12].

From Van de Vel [12] we have the following theorems.

Theorem: 5.1 For a surjective C P function $f: X \rightarrow Y$ the following are true.

(1) If $h(X) \geq h(Y)$ and $r(X) \geq r(Y)$

(2) If f is also C C then $C(X) = C(Y)$ and $c(X) \geq c(Y)$

Theorem 5.2. The following are equivalent for any convex structure

(1) If $h(X) \leq 3$ and if X is S_3 Then X is S_4.

(2) If $h(X) \leq 2$, and if X is S_2 then X is S_4.

Theorem 5.3. Let V be a finite dimensional vector space over the totally ordered field K, and let C be the

H-convexity on V generated symmetrically by a set $\mathcal{F}$ of linear functionals. If $\mathcal{F}$ is finite or if $K = R$, then, $h(V,\mathscr{C}) = md(\mathcal{F})$ where $md(\mathcal{F}) = \sup\{|\mathcal{F}_0|:\mathcal{F}_0 \subseteq \mathcal{F}$ and $\mathcal{F}_0$ is minimaly dependent$\}$ is the degree of minimal dependence of $\mathcal{F}$.

We also have the following,

Theorem 5.4 [8]. Suppose H is a subset of R^n. Then H is a hyperplane if and only if there exists a non identically zero linear functionals f and a real constant δ such that $H = f^{-1}(\delta) = \{x \in R^n: f(x) = \delta\}$.

From these theorems, the following observations can be made.

1) The Helly number of any H-convexity on R^n is at most n+1

2) Any symmetric H-convexity on R^2 is S_4.

3) If $\mathcal{F}$ is a collection of linear functionals corresponding to a family of planes in R^3 whose intersection is a singleton and $|\mathcal{F}| \geq 4$, then the Helly number of the symmetrically generated H-convexity is 4.

5.2 A PROBLEM OF VAN DE VEL

In this section we consider a problem of

Van de Vel [12] and obtain some interesting results of the symmetrically generated H-convexity of R^3.

PROBLEM: Is each symmetric H-convexity of arity 2 ?

We studied the above problem and give an example of a symmetric H-convexity of infinite arity. We get a sufficient condition under which a family of linear functionals generates a symmetric H-convexity of arity 2.

Consider the vector space R^3 over R and let $\mathcal{F}$ be any collections of linear functionals over R^3. Let $\mathcal{E}$ be the H-convexity generated by $\mathcal{F}$. Then, for any $x_1, x_2 \in R^3$.

$$Co\{x_1, x_2\} = \bigcap \{f^{-1}[f(x_1), f(x_2)] : f \in \mathcal{F}\}.$$

By $[f(x_1), f(x_2)]$ we mean the set of all convex combinations of $f(x_1)$ and $f(x_2)$.

By theorem 5.4 each linear functional on R^3 corresponds to a plane in R^3. Now we prove,

Theorem 5.5. Let $\mathcal{F}$ be a family of linear functionals corresponding to a family of planes intersecting in a line,

then the arity of the H-convexity symmetrically generated by $\mathcal{F}$ is two.

proof: Let $C \subset R^3$ have the property that $Co\{x_1,x_2\} \subset C$ whenever $x_1,x_2 \in C$. To prove that C is convex. Let $F \subset C$ where $|F| > 2$ and let $y \in Co(F)$. Let $f \in \mathcal{F}$. Then,

Claim: There are $x_1,x_2 \in F$ such that
$$f(x_1) \leq f(y) \leq f(x_2).$$

Otherwise, if $f(y) < f(x)$ for each $x \in F$ or $f(y) > f(x)$ for each x in F, then, $f^{-1}(-\infty, f(y)]$ or $f^{-1}[f(y) \infty)$ will be a half pace containing y and not intersecting with F. So $y \notin Co(F)$. Hence the claim.

Therefore, for each $f \in \mathcal{F}$, $f^{-1}(f(y))$ meets the standard convex hull of F. Since, $y \in f^{-1}(f(y))$ for each $f \in \mathcal{F}$, $\cap \{f^{-1}(f(y)): f \in \mathcal{F}\} \neq \phi$. Now, because $\mathcal{F}$ corresponds to the family of planes intersecting in a straight line, the set $\cap \{f^{-1}(f(y): f \in \mathcal{F}\}$ is a straight line. Let f and g be such that the angle between $f^{-1}(f(y))$ and $g^{-1}(g(y))$ is the maximum (see fig 5.3)
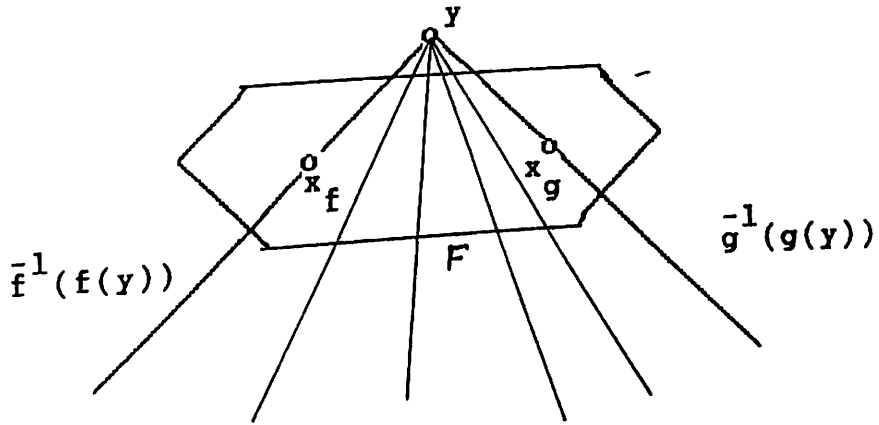
$$f^{-1}(f(y)) \qquad F \qquad g^{-1}(g(y))$$

Fig. 5.3

Let $x_f \in f^{-1} f(y) \cap F_c$ and $x_g \in g^{-1}(g(y)) \cap F_c$ where F_c is the standard convex hull of F. Then $y \in Co(\{x_f, x_g\}) \subset C$. Hence $Co(F) \subset C$ and therefore C is convex. Hence the H-convexity generated by $\mathcal{F}$ is of arity 2. $\qquad \square$

The above theorem is not true for a family of functionals corresponding to a family of planes whose intersection is a singleton. The following example gives an example of a symmetrically generated H-convexity of infinite arity.

Let F be the linear functionals corresponding to the tangent planes of a cone, whose cross section is a circle parallel to the x-y plane. That is, $f \in \mathcal{F}$ corresponds to the planes making a constant angle with the

x-y plane.   Let us assume that this angle is $\pi/4$. That is,

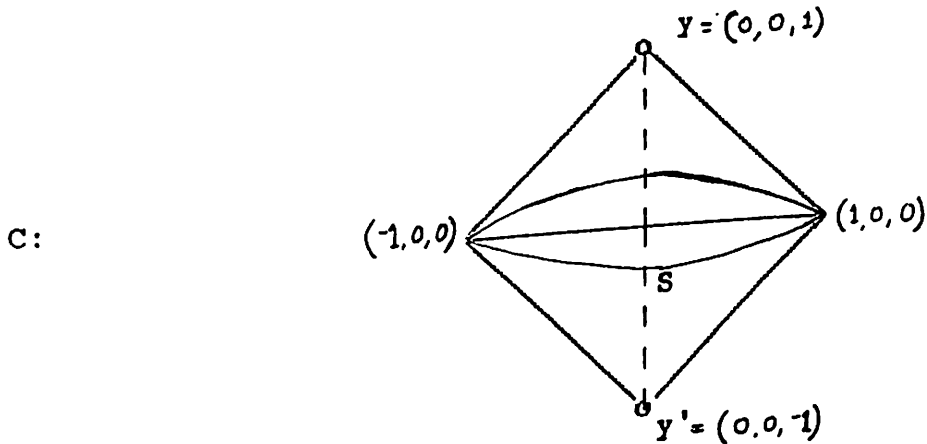$$\mathcal{F} = \{f: f(x,y,z) = y \cos \alpha - x \sin \alpha - z, \ \alpha \in [0,2\pi)\}$$

C:

Fig. 5.4

Now the solid C which is the convex hull of S (See Fig 5.4)

is a convex set.

Let $C_1 = C\backslash\{y,y'\}$.

It is clear that $y \in Co(C_1)$.   Also C_1 is convex with respect

to the standard convexity.

That is $f^{-1}(f(y)) \cap C_1 \neq \phi$ for each $f \in \mathcal{F}$.

But note that $f^{-1} f(y) \cap g^{-1}(g(y)) \cap C_1 = \phi$ if $f \neq g$.

hence corresponding to each f, we get $x_f \in C_1$

Such that $x_f \neq x_g$ whenever $f \neq g$. Now, since $\mathcal{F}$ is infinite

$\{x_f: f \in \mathcal{F}\}$ is infinite.

Hence C_1 is with the property that $Co(F) \subset C_1$ for each finite set contained in C_1 but C_1 is not convex. Hence the convexity generated by $\mathscr{F}$ is of infinite arity. Further, it is of uncountable arity.

Remark 5.1. a). Since the above H-convexity is of arity greater than 2, it is not JHC.

b). For any n, if we replace the cone whose cross section is a circle by a pyramid whose <u>crossection</u> is a regular 2n-gon, the H-convexity symmetrically genetated by the family of functionals corresponding to the family of tangent planes containing the lateral faces, is of arity n.

Remark 5.2. R^3 with the H-convexity generated by the family of functionals corresponding to the tangent planes of a cone, doesn't have the Peano property. For, let

$\mathscr{F} = \{f: f(x,y,z) = y \cos \alpha - x \sin \alpha - z, \alpha \in [0,2\pi)\}$.

Let $a = (-1,0,0)$, $b = (0,0,1)$, $c = (1/2, 0, 1/2)$ and $u = (1/2,0,0)$, $v = (1/2,1/4,1/4)$

Then $u \in Co(\{a,b\})$. Also note that $v \in Co(\{c,u\})$

(See fig 5.5)
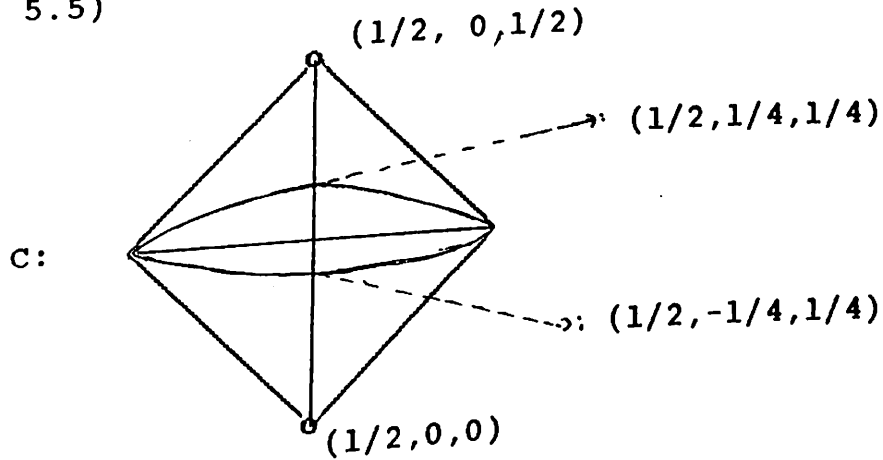

Fig. 5.5

Note that $Co(\{c,u\})$ is the solid in fig 5.2, because any plane P making an angle $\pi/4$ with the x-y plane will either cut the ordinary segment cu or the solid C will be contained in one of the half spaces determined by P.
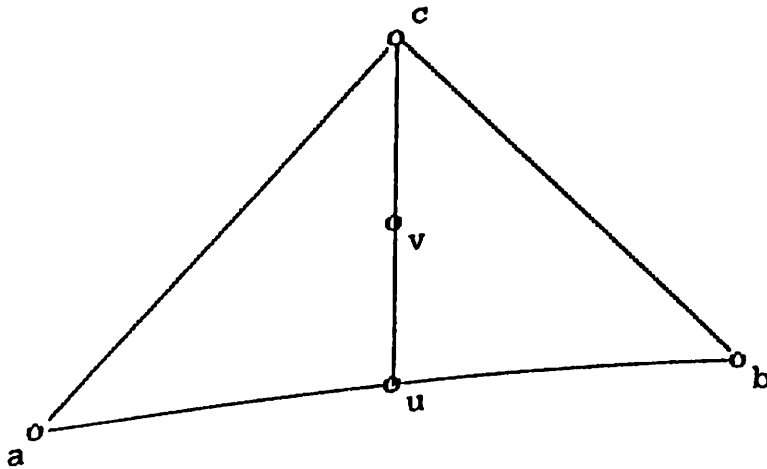

Fig 5.6

Define f: on R^3 as $f(x,y,z) = x-z$. Then $f \in \mathcal{F}$.

Then $f(x,y,z) \leq 0$ is a half plane containing both a and c.

But for $v = (1/2, 1/4, 1/4)$, $x-z > 0$.

Hence $v \notin Co(\{a,c\})$.

Note that $Co(\{b,c\}) = bc$, the ordinary segment

joining b and c, because it is the intersection of the solid

C, the plane $x+z = 0$, and the convex set

$C_0 = \{(x,y,z) : 0 \leq x-z \leq 1\}$.

Now for any $v^1 \neq c$ in $Co\{b,c\}$, .

Let $v^1 = (x_0, y_0, z_0)$. Then $x_0 > 1/2$ and $z_0 < 1/2$.

In this case, $y_0 + z_0 < 1/2$.

Define $g: R^3 \to R$ such that $g(x,y,z) = y + z$. Here $g \in \mathcal{F}$.

Then, the half space H: $f(x,y,z) < 1/2$ contain both v^1 and a

but $v \notin H$.

Hence $v \notin Co(\{a, v^1\})$.

Remark 5.3. The H-convexity defined in the example is not

S_4. For, the sets $\{(x,y,z) : z = 0, y = 0\}$ and

$\{(x,y,z): z = 1, x = 0\}$ are convex sets which can not be

separated.

Now we give a characterization for an H-convexity

in R^3 to be S_4.

Theorem 5.6: The H-convexity symmetrically generated by a family of linear functionals $\mathcal{F}$ is S_4 if and only if for any two intersecting convex straight lines, the plane determined by these lines is convex. That is, $\mathcal{F}$ should contain the functionals corresponding to the plane determined by these lines.

Proof: Let ℓ_1 and ℓ_2 be any two intersecting convex lines. Then, ℓ_1 can be separated from a line ℓ which is parallel to ℓ_2 and which does not intersect with ℓ_1, only by a plane containing ℓ_1 and ℓ_2.

Now let $\mathcal{C}$ be the H-convexity on R^3 with the given condition and let C_1 and C_2 be disjoint convex sets. Since C_1 and C_2 are determined by half spaces, there are half spaces

$H_1, H_2, \ldots, K_1, \ldots$ such that,

$$C_1 = H_1 \cap H_2 \cap \cdots \quad \text{and}$$

$$C_2 = K_1 \cap K_2 \cap \cdots$$

$$C_1 \cap C_2 = H_1 \cap H_2 \cap \cdots \cap K_1 \cap \cdots \cap K_m = \phi$$

Now, since the Helly number is at most four, the intersection of some four membered subfamily of the above family of half spaces is empty (see [12])

$$\text{If } H_i \cap K_{i,1} \cap K_{i,2} \cap K_{i,3} = \phi$$

Then $H_i \cap C_2 = \phi$ and $C_1 \subset H_i$ and H_i is the required half space.

$$\text{If } H_i \cap H_j \cap K_k \cap K_\ell = \phi \text{ , let } P_i, P_j, P_k, \text{ and } P_\ell \text{ be the}$$

corresponding planes.

Let $\ell_{i,j} = P_i \cap P_j$ and $\ell_{k,\ell} = P_k \cap P_\ell$.

Let P_0 be the plane determined by $\ell_{i,j}$ and the line ℓ which intersect with $\ell_{i,j}$ and which is parallel to $\ell_{k,\ell}$. Then P_0 separates C_1 and C_2. □

Now the following example gives an H-convexity on R^3 which satisfies both Pasch and Peano properties but is neither JHC nor S_4.

Let $\mathcal{F} = \{f: f(x,y,z) = \tan\theta\,(y\cos\alpha - x\sin\alpha) - z:$

$\alpha \in [0,2\pi),\ \theta \in [\pi/4, \pi/2)\} \cup \{f: f(x,y,z) = ax+by,$

$a,b \in R\}.$

Then we observe that the H-convexity symmetrically generated

by $\mathcal{F}$ has the following properties.

Property 1. Each straight line in R^3 is convex.

For this we prove that any straight line is contained in two

distinct convex planes. If ℓ is perpendicular to the x-y

plane, it is trivially true. Actually there are infinite

number of convex planes by the choice of $\mathcal{F}$. Now for any ℓ,

there is a plane perpendicular to the x-y plane, which

contains ℓ. Assume without loss of generality that ℓ passes

through $(0,0,0)$. Then for any $(x_1, y_1, z_1) \in \ell \setminus \{(0,0,0)\}$.

Then $y_1 x - x_1 y = 0$ is a plane perpendicular to the x-y plane

and containing ℓ.

Now if $\pi/4 \leq \theta < \pi/2$, then, by the choice of $\mathcal{F}$ we get an α

such that, the plane,

$\tan\theta\,(y\cos\alpha - x\sin\alpha) - z = 0$, will contain ℓ.

Now let $0 \leq \theta \leq \pi/2$. Assume without loss of generality that the plane perpendicular to the x-y plane which contain ℓ is the x-z plane.

Let $(h,0,h+k) \in \ell$, where $k > 0$. Then

Let $\alpha = \sin^{-1}(-h/h+k)$. Then,

$y \cos \alpha - x \sin \alpha - z = 0$ is a convex plane containing ℓ.

Hence each straight line is convex.

Property 2. This is a Pasch- Peano space.

For any a,b,c,u,v such that, $u \in a\,b$, $v \in c\,u$ we get a v^1 on $b\,c$ such that $v \in a\,v^1$. This is because the convex hull of any two points is the ordinary segment joining those points. So this is having the Peano property. Using similar arguments we can prove that it is having the Pasch property. But this is neither JHc nor S_4, because any line on the x-y plane is convex but the plane is not convex. Therefore by theorems 1.1 and 1.2 this convexity is not of arity two.

5.3 CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER STUDY.

This thesis is an attempt to find out some properties of d.c.s. graphs, m.c.s. graphs, interval

monotone graphs and totally non-interval monotone graphs. We have also introduced a new type of convexity to the edge set of graphs and its convex invariants and Pasch Peano properties are analysed. Also we discuss some properties of H-convexity.

The results of this thesis are far from being complete. We list some of the problems which we have either not attempted or found the answers to be difficult.

1. Characterize solvable trees.

2. Determine the size of the smallest d.c.s. graph containing a nonsolvable tree. Equivalently is it possible to express the size of the smallest d.c.s. graph containing any tree as a function of the order, diameter, radius and the degree?.

3. In the corollary of Theorem 2.14, is it possible to replace $K_{n,n}$ by any m.c.s. graph G of sufficiently large size with the property that $I(a,b) \neq V(G)$ for any pair $a,b \in V(G)$?

4. Characterize halfspce free graphs.

5. Characterize JHC graphs.

6. Since the study of edge convexity has been just initiated, properties of convexity in V(G) studied in detail by many authors can be attempted in this case also.

7. Characterize the H-convexity of arity two.

8. Characterize S_i graphs for i = 2,3 and 4.