**C 30306** (Pages : 4) Name.....................................

Reg. No.....................................

# FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2017

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

## Section A

*Answer* **all** *the twelve questions.*
*Each question carries* 1 *mark.*

1. True *or* False : Every binary operation on a set consisting of a single element is both commutative and associative.,

2. Describe the isomorphism from $<\mathbb{Z},+>$ into $<n\mathbb{Z},+>$.

3. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.

4. Determine whether the set of all invertible $n \times n$ real matrices with determinant $-1$ is a subgroup of GL $(n, \mathbb{R})$.

5. Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is a permutation of $\mathbb{R}$.

6. Write the orbits of the identity permutation, $i$ on a set A

7. True *or* false : Every finite group contains an element of every order that divides the order of the group.

8. Find all orbits of the permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$.

9. Find all units in the ring $\mathbb{Z}_4$.

10. Find the characteristics of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$.

11. How many solutions, does the equation $x^2 - 5x + 6 - 0$ have $\mathbb{Z}_7$ ?

12. Find the number of elements in the set $\{\sigma \in S_5 / \sigma(2) = 5\}$ ?

(12 × 1 = 12 marks)

**Turn over**

## Section B

*Answer any* **ten** *out of fourteen questions.*
*Each question carries* 4 *marks.*

13. Define isomorphism of algebraic structures. Determine whether the function $\phi : < \mathbb{Q}, + > \to < \mathbb{Q}, + >$ where $\phi(x) = \frac{x}{2}, x \in \mathbb{Q}$ is an isomorphism.

14. Define a group. Is the set $\mathbb{Q}^+$ of positive rational numbers under multiplication a group. Justify your answer ?

15. Let G be a group and a be a fixed element of G. Show that $H_a = \{x \in G : xa = ax\}$ is a subgroup of G.

16. Prove that an infinite cyclic group has exactly two generators.

17. Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$ as a product of disjoint cycles and then as a product of transpositions.

18. Find all generators of $\mathbb{Z}_6, \mathbb{Z}_8$ and $\mathbb{Z}_{20}$.

19. If A is any set and $\sigma$ is a permutation of A, show that the relation '$\sim$' defined on A by $a \sim b$ if and only if $b = \sigma^n(a)$, for some $n \in \mathbb{Z}, a, b \in A$ is an equivalence relation.

20. Let H be a subgroup of a group G, and let $a \in G$. Define the left and right cosets of H containing $a$. Exhibit all left and right cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

21. Prove that a group homomorphism $\phi : G \to G'$ is a one-one map if and only if $\text{Ker}(\phi) = \{e\}$.

22. Define Ring. Give an example.

23. Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$ in $\mathbb{Z}_{12}$.

24. Define characteristic of a ring. Find the characteristic of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$ and $\mathbb{Z}_3 \times \mathbb{Z}_4$.

25. Show that the intersection of two normal subgroups of a group is a normal subgroup.

26. If $\mathbb{R}$ is a ring such that $a^2 = a \in \mathbb{R}$. Prove that $\mathbb{R}$ is a commutative ring.

(10 × 4 = 40 marks)

## Section C

*Answer any* **six** *out of nine questions.*
*Each question carries 7 marks.*

27. Let G be a set consisting of ordered pairs $(a, b)$ such that $a, b$ are real and $a \neq 0$. A binary operation $*$ is defined on G by $(a, b) * (c, d) = (ac, bc + d)$. Prove that G is a group under $*$. Is G abelian ? Justify.

28. Show that a nonempty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H, \forall\, a, b \in H$.

29. Let G and $G'$ be groups and let $\phi : G \to G'$ be a one-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Prove that $\phi[G]$ is a subgroup of $G'$ and $\phi$ is an isomorphism of G with $\phi[G]$.

30. Define abelian group. Prove that a group G is abelian if every element except the identity e is of order 2.

31. Show that the set of all permutations on three symbols forms a finite non-abelian group $S_3$ of order 6 with respect to permutation multiplication.

32. Let $\phi : G \to G'$ be a group homomorphism and let $H = \text{Ker}(\phi)$. For $a \in G$, prove that the set $\{x \in G / \phi(x) = \phi(a)\}$ is the left coset $aH$ of H.

33. Show that $a^2 - b^2 = (a + b)(a - b)$ for all $a, b$ in a ring $\mathbb{R}$ if and only if $\mathbb{R}$ is commutative.

34. Show that the characteristics of an integral domain D must be either 0 or a prime $p$.

35. Describe the field of quotients of an integral domain.

(6 × 7 = 42 marks)

## Section D

*Answer any* **two** *out of three questions.*
*Each question carries 13 marks.*

36. (a) Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G. (6 marks)

(b) Prove that a subgroup H of a group G is normal in G if and only if each left coset of H in G is a right coset of H in G. (7 marks)

**Turn over**

37. (a) Prove that no permutation in $S_n$ can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions. (7 marks)

(b) Prove that a subgroup of a cycle group is cyclic. (6 marks)

38. (a) Show that the order of an element of a finite group divides the order of the group.

(7 marks)

(b) Prove that every finite integral domain is a field.

(6 marks)

[2 × 13 = 26 marks]

C 30306-A (Pages : 4) Name..................................

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# FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2017

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

(Multiple Choice Questions for SDE Candidates)

| Time : 15 Minutes | Total No. of Questions : 20 | Maximum : 20 Marks |
| --- | --- | --- |

## INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. The candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer-book.
4. The MCQ question paper will be supplied after the completion of the descriptive examination.

# MAT 5B 06—ABSTRACT ALGEBRA

(Multiple Choice Questions for SDE Candidates)

1. Which of the following is a non-abelian group with the property that each proper subgroup is abelian ?

   (A) $S_3$.   (B) $Z_8$.

   (C) $S_5$.   (D) None of these.

2. Let G be a cyclic group of order 6. Then the number of $g \in G$ such that $G = \langle g \rangle$ is :

   (A) 5.   (B) 3.

   (C) 2.   (D) 4.

3. If $a$, $b$ are elements of a group G of order $m$ then order of $ab$ and $ba$ are :

   (A) Same.   (B) Equal to $m$.

   (C) Unequal.   (D) None of these.

4. How many elements are there on the cyclic subgroup of Z30 generated by 25 ?

   (A) 3.   (B) 6.

   (C) 5.   (D) None of these.

5. If $a$ is a generator of a cyclic group G, then :

   (A) $a^2$ is a generator.   (B) $a^{-1}$ is a generator.

   (C) G has no other generators.   (D) Every subgroup of G is generated by $a$.

6. The set of real numbers the operation addition is ————.

   (A) Not a group.   (B) Abelian but not cyclic.

   (C) Not Abelian but cyclic.   (D) A group but not abelian.

7. Let G be a group and let $a * b * c = e$ for $a, b, c \in G$. Then $b * c * a$ equals :

   (A) $a$.   (B) $e$.

   (C) $b$.   (D) None of these.

8. Which of the following set, addition is not a binary operation ?

(A) Complex numbers. (B) Real numbers.

(C) Non zero real numbers. (D) Integers.

9. Which of the following are true ?

(1) Group may have more than one identity element.

(2) Any two groups of three elements are isomorphic.

(3) Every group of at most three elements is abelian.

(A) 2 and 3. (B) 1 and 2.

(C) 1 and 3. (D) All.

10. The order of 2 in the group (Z, + ) .where Z is the set of integers is ————.

(A) 0. (B) 4.

(C) Not defined. (D) Infinity.

11. Which of the following is an even permutation ?

(A) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 1 & 6 & 2 & 1 & 8 \end{pmatrix}$. (B) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 5 & 3 & 7 & 8 & 6 \end{pmatrix}$.

(C) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 3 & 5 & 2 & 6 & 8 & 7 \end{pmatrix}$. (D) None of these.

12. Cayley's theorem is :

(A) Order of the subgroup divides the order of the group.

(B) Every group is isomorphic to a group of permutations.

(C) Every group is isomorphic some cyclic group.

(D) Every group of prime order is cyclic.

13. The number of cosets of the subgroup 3Z in (Z, +) is ————.

(A) 2. (B) 3.

(C) Infinite. (D) Cannot be determined.

14. The cyclic subgroup of $Z_{24}$ generated by 18 has order ————.

(A) 4. (B) 6.

(C) 9. (D) None of these.

15. In a non-abelian group the element a has order 108. Then the order of a 12 is :

(A) 54. (B) 27.

(C) 18. (D) 9.

16. Which of the following is true ?

(A) Every homomorphism is a one to one map.

(B) A homomorphism may have an empty kernel.

(C) For any two groups G and K, there exists a homomorphism of G into K.

(D) For any two groups G and K, there exists an isomorphism of G onto K.

17. If G is an infinite cyclic group, then how many generators are there for the group G ?

(A) Exactly two. (B) At least two.

(C) Infinitely many. (D) Only one.

18. The set of non zero real numbers the operation addition is ————.

(A) Not a group. (B) Abelian but not cyclic.

(C) Not Abelian but cyclic. (D) A group but not abelian.

19. The set $M_2$ (R) of all 2 × 2 matrices with real entries have ————.

(A) No zero divisors. (B) Only two Zero devisors.

(C) Infinite number of zero divisors. (D) None of these.

20. Let Z be the set of integers, then (Z, +, × ) is not a ————.

(A) Field. (B) Integral domain.

(C) Commutative ring. (D) None of these.