C 30306 **(Pages : 4)** **Name..**

Reg. No.....................................

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2017

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all** *the twelve questions.*
Each question carries 1 *mark.*

1. True *or* False : Every binary operation on a set consisting of a single element is both commutative and associative.,

2. Describe the isomorphism from $<\mathbb{Z},+>$ into$<n\mathbb{Z},+>$.

3. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.

4. Determine whether the set of all invertible $n \times n$ real matrices with determinant -1 is a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

5. Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is a permutation of $\mathbb{R}$.

6. Write the orbits of the identity permutation, i on a set A

7. True *or* false : Every finite group contains an element of every order that divides the order of the group.

8. Find all orbits of the permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$.

9. Find all units in the ring $\mathbb{Z}_4$.

10. Find the characteristics of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$.

11. How many solutions, does the equation $x^2 - 5x + 6 - 0$ have $\mathbb{Z}_7$?

12. Find the number of elements in the set $\{\sigma \in S_5 / \sigma(2) = 5\}$?

(12 × 1 = 12 marks)

Turn over

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 *marks.*

13. Define isomorphism of algebraic structures. Determine whether the function $\phi : < \mathbb{Q}, + > \to < \mathbb{Q}, + >$ where $\phi(x) = \frac{x}{2}, x \in \mathbb{Q}$ is an isomorphism.

14. Define a group. Is the set $\mathbb{Q}^+$ of positive rational numbers under multiplication a group. Justify your answer ?

15. Let G be a group and a be a fixed element of G. Show that $H_a = \{x \in G : xa = ax\}$ is a subgroup of G.

16. Prove that an infinite cyclic group has exactly two generators.

17. Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$ as a product of disjoint cycles and then as a product of transpositions.

18. Find all generators of $\mathbb{Z}_6, \mathbb{Z}_8$ and $\mathbb{Z}_{20}$.

19. If A is any set and σ is a permutation of A, show that the relation '$\sim$' defined on A by $a \sim b$ if and only if $b = \sigma^n(a)$, for some $n \in \mathbb{Z}, a, b \in A$ is an equivalence relation.

20. Let H be a subgroup of a group G, and let $a \in G$. Define the left and right cosets of H containing a. Exhibit all left and right cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

21. Prove that a group homomorphism $\phi : G \to G'$ is a one-one map if and only if $\text{Ker}(\phi) = \{e\}$.

22. Define Ring. Give an example.

23. Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$ in $\mathbb{Z}_{12}$.

24. Define characteristic of a ring. Find the characteristic of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$ and $\mathbb{Z}_3 \times \mathbb{Z}_4$.

25. Show that the intersection of two normal subgroups of a group is a normal subgroup.

26. If $\mathbb{R}$ is a ring such that $a^2 = a \in \mathbb{R}$. Prove that $\mathbb{R}$ is a commutative ring.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

27. Let G be a set consisting of ordered pairs (a, b) such that a, b are real and $a \neq 0$. A binary operation $*$ is defined on G by $(a, b) * (c, d) = (ac, bc + d)$. Prove that G is a group under $*$. Is G abelian ? Justify.

28. Show that a nonempty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H, \forall\, a, b \in H$.

29. Let G and G' be groups and let $\phi : G \to G'$ be a one-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Prove that $\phi[G]$ is a subgroup of G' and ϕ is an isomorphism of G with $\phi[G]$.

30. Define abelian group. Prove that a group G is abelian if every element except the identity e is of order 2.

31. Show that the set of all permutations on three symbols forms a finite non-abelian group S_3 of order 6 with respect to permutation multiplication.

32. Let $\phi : G \to G'$ be a group homomorphism and let $H = \text{Ker}(\phi)$. For $a \in G$, prove that the set $\{x \in G \,/\, \phi(x) = \phi(a)\}$ is the left coset aH of H.

33. Show that $a^2 - b^2 = (a + b)(a - b)$ for all a, b in a ring $\mathbb{R}$ if and only if $\mathbb{R}$ is commutative.

34. Show that the characteristics of an integral domain D must be either 0 or a prime p.

35. Describe the field of quotients of an integral domain.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 marks.

36. (a) Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G. (6 marks)

    (b) Prove that a subgroup H of a group G is normal in G if and only if each left coset of H in G is a right coset of H in G. (7 marks)

Turn over

37. (a) Prove that no permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions. (7 marks)

(b) Prove that a subgroup of a cycle group is cyclic. (6 marks)

38. (a) Show that the order of an element of a finite group divides the order of the group.

(7 marks)

(b) Prove that every finite integral domain is a field.

(6 marks)

[2 × 13 = 26 marks]