D 40044 (Pages : 3) Name.......................................

Reg. No..................................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH/APRIL 2018

(CUCBCSS—UG)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. Define gcd of two integers.
2. Find lcm (– 15, 20).
3. Define a Diophantine equation in two variables.
4. Write the canonical form of 180.
5. State Wilson's theorem.
6. Define a pseudoprime.
7. Find $\phi(9)$.
8. Define subspace of a vector space.
9. Give a spanning subset of the vector space of all polynomial functions over $\mathbb{R}$.
10. Show that any set of vectors which contains the zero vector is linearly dependent.
11. Define a linear map.
12. Define kernel of a linear map.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of* **fourteen** *questions.*
Each question carries 4 *marks.*

13. Show that $\frac{a\left(a^2+2\right)}{3}$ is a positive integer for any positive integer a.
14. Prove that two non-zero integers a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.
15. Find gcd (1769, 2378).

Turn over

16. Is $\sqrt{2}$ a rational number ? Justify your answer.

17. Find the remainder when 41^{65} is divided by 7.

18. Define an absolute pseudoprime. Illustrate with an example.

19. If p and q are primes, show that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

20. If n is a squarefree positive integer, prove that the positive divisors of n is 2^r, where r is the number of positive divisors of n.

21. Show that for a positive integer r, the product of any r consecutive positive integers is divisible by $r!$.

22. Define a vector space.

23. Prove that a non-empty subset W of a vector space V over a field F is a subspace of V if and only if $c\alpha + \beta \in W$ for all $\alpha, \beta \in W$ and for all $c \in F$.

24. Show that the set $\{e_1, e_2, e_3, e_4\}$, where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$, is a basis of $\mathbb{R}^4$.

25. Show that the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(a, b) = (a + b, a - b)$ is linear.

26. If V is a vector space of dimension $n \geq 1$ over a field F, show that V is isomorphic to the vector space F^n.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of* **nine** *questions.*
Each question carries 7 marks.

27. Given integers a and b, not both of which are zero, prove that there exist integers x and y such that $\gcd(a, b) = ax + by$.

28. Prove that the linear Diophantine equation $ax + by = c$ has a solution if and only if $d | c$ where $d = \gcd(a, b)$. Verify whether the Diophantine equation $14x + 35y = 93$ can be solved.

29. Solve the system of congruences $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$.

30. State and prove Fermat's Little Theorem.

31. Obtain the number and sum of positive divisors of a positive integer n.

32. Are the intersection and union of two subspaces of a vector space V again subspaces of V? Justify your answer.

33. A non-empty subset S of a vector space V is a basis of V if and only if every element of V can be expressed in a unique way as a linear combination of elements of S.

34. Show that the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$ is not surjective.

35. Let V and W be vector spaces over a field F. If the set $\{v_1, v_2, v_3, \ldots, v_n\}$ is a basis of V and if $w_1, w_2, w_3, \ldots, w_n$ are elements of W, prove that there is a unique linear mapping $f : V \to W$ such that $f(v_i) = w_i$ $(i = 1, 2, 3, \ldots, n)$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of* **three** *questions.*
Each question carries 13 *marks.*

36. Prove that if a and b are integers with $b \neq 0$, then there exist unique integers q and r such that $a = qh + r,\ 0 \leq r \leq |b|$.

37. Show that Euler's phi-function is multiplicative.

38. If V and W are vector spaces over a field F and if $f : V \to W$ is a linear map, prove that :

(a) $f(v_1 - v_2) = f(v_1) - f(v_2)$

(b) the set Ker $f = \{v \in V : f(v) = 0\}$ is a subspace of V.

(c) for any subspace X of V, $f(X)$ is a subspace of W.

(d) f is an isomorphism if and only if Ker $f = \{0\}$.

(2 × 13 = 26 marks)