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# [GTN XXXIV:4] (k,2)-CONVEX GRAPHS: MINIMAL PATH CONVEXITY

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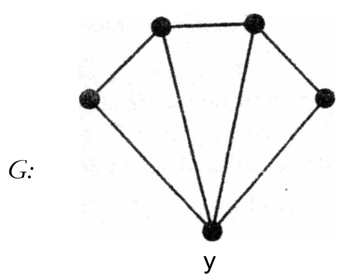
**Abstract**
A necessary and sufficient condition for the existence of a 2-connected (k,2)-convex graph is obtained.

## 1. Introduction

In this paper we consider only finite, undirected graphs. We follow the notation and terminology of Buck] and Harary [1]. Convexity can be defined for the vertex set of a graph $G = (V, E)$ in several ways. Pro inently, A $\subseteq$ V is *convex* if for all x, y in A, {z : z is in some $xy$ geodesic $\subseteq A$ ; and it is *minimal path cc vex* (m-convex) if {z : z is in some chordless $xy$ path } $\subseteq$ A where a chordless $xy$ path is a path $P: x = x_1$, $x_n$ = y such that no nonconsecutive vertices in $P$ are adjacent in $G$. Since any shortest path is chordless is obvious that an m-convex subset of V is convex. An example of a convex set that is not m-convex is sho in Figure 1. Several aspects of these types of convexities have been studied by Mulder [2], Farber [3], Dud [4], and Bandelt [5]. An excellent survey has been made by Van de Vel [6].



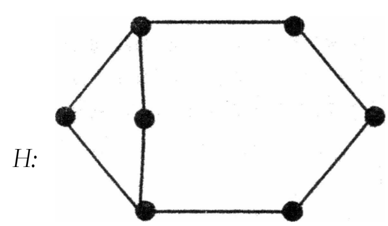




Figure 1: $\{x, y, z\}$ is convex but not m-convex

Figure 2

In a graph $G$, the empty set, singletons, sets inducing complete subgraphs, and $V(G)$ are trivially convex. an attempt to classify graphs according to the number of nontrivial convex sets, Hebbare [7] called a gra $(k,\omega)$-convex if it has exactly $k$ nontrivial convex sets and its clique number, the order of the largest cliq is w. If $k = 1$ , the graph is said to be *uniconvex* [8]. (0,2)-convex graphs are called *convex simple*, if c vexity is considered, or m-convex simple, if m-convexity is considered. Although every convex simple gra is m-convex simple, the converse is not true (see Figure 2). Moreover, C, for $n > 4$ and $Q_n$ for $n > 2$ are convex simple but are not convex simple.

It is known [9] that, for any given $k$ a (k,2)-convex graph can he constructed for convexity. Here, we consi only m-convexity and make some observations.

## 2. Main Results

In a connected graph $G$, $S \subseteq V$ is called a *separator* if $G \backslash S$ is disconnected. Further, it is a *clique separa* if $\langle S \rangle$ is a clique.

Theorem 2.1 [10]: A graph $G$ is m-convex simple if and only if it has neither a nontrivial clique nor a clique separator. U

**Definition 2.1:** A convex set in which no proper subset with cardinality at least three is convex, is called a *minimal nontrivial convex set.* ■

**Theorem 2.2:** *Let* $k$ be a positive integer. The following are equivalent:

1) There exists a 2-connected (k,2)-convex graph.
2) There is a block graph H such that $k = |H_c|$, $H = \{C \subset V(H) : C$ is convex, $C \neq 0$, VI

**Proof:**

1) 2): Let $G$ be a 2-connected (k,2)-convex graph, and let $C_1, C_2, \ldots, C_n$ be the minimal convex su V. Let $C_i \cap C$ $S_i$ Clearly, $|S_{ij}|$ 5 2 for $i$ $j$ and since $S_{ij}$ is convex $\langle S_{ij}\rangle$ is a clique provide nonempty.

<u>Claim 1:</u> $S_u$ is a clique separator if and only if 1;1 = 2.

Let $S_{ij} = \{x, y\}$. Since $S_i$ is convex, x is adjacent toy.

To prove that $S_u$ is a separator for $G$, suppose otherwise. Then each vertex of $C_i\backslash S_{ij}$ is connected to in $C_j\backslash S_{ij}$ by some path in $G\backslash S_{ij}$. Let $c_i$ E $C_i$ $c_j$ E $C_j$, and let $c_0 u_p u_2, \ldots, u_l, c_j$ be a chordless $c_i$ in $G\backslash S_j$ Assume without loss of generality that $u_k \notin C_i$ for $k = 1, 2, \ldots, l$. Since $G$ is triangle free, adjacent to at least one vertex, say $x$ of $S_{ip}$ Note that $c_j$ and x are connected by some path. Thus, th chordless $c_jx$ path that contains vertices not in $C_i$. This contradicts the fact that $C_i$ is convex. Hence 4 disconnected. The converse is obvious because $G$ *is* 2-connected.

Now form the graph $H$ with $V(H) = \{C_1, C_2, \ldots, C_n\}$ and $C_i$ adjacent to $C_j$ if $C_i \cap C_j$ is a clique se

<u>Claim 2:</u> $H$ is a block graph.

If not, there will be a block $B$ *in* $H$ and $C_i, C_j$ E $B$ such that $C_i$ is not adjacent to $C_j$ in $H$. Assume with of generality that $d(C_i,$ = 2 and let $i = 1$, $j = 3$. Let $C_1, C_2, C_3$ be a path and since $B$ is a bloc will be another chordless path $C_3, C_4, \ldots, C_t, C_1$ connecting C3 and C1.

Let $C_i \cap C_{i+1} = S_i$ for $i = 1, 2,$ $t$, where + denotes addition modulo t. Now $S_1 * S_2$, $d(C_i, C_j) = 2$. Since G is triangle free, there exists x E $S_1$, y E $S_2$ such that $xy \notin E(G)$. Note that x, Now, since $C_i \cap C_{i+1} \neq \emptyset$, each $C_i$ induces a 2-connected triangle-free subgraph, $x E$ $C_1$ and Thus, we obtain a chordless $xy$ path containing vertices not in C2, a contradiction.

Note that $C_1, C_2,$ $C_n$ have the following properties.

1) If $C_i \cap C_j$ is a clique separator, then $C_i \cup C_j$ is convex.
2) If $C_i \cap C_j$, $C_j \cap C_k$, $C_k \cap C_i$ are clique separators, then $C_i \cap C_j = C_j \cap C_k = C_k \cap C_i$.
3) If $d(C_i) = a$ and if the neighbors of $C_i$ are pairwise nonadjacent, then $(C_i)$ has at least $a$ edges

Hence it follows that, the convex subsets of $G$ correspond to the connected subsets of $H$, which are p the convex subsets.

2) 1): Let $H$ be a block graph. We can construct a triangle-free graph $G'$ so that the convex subset correspond to the connected subsets of $H$. This can be done by replacing each vertex $u$ of $H$ by a co an m-convex simple graph $G$ of sufficiently large size in such a way that $G_{u_1} \cap G_{u_2}$ only if $u_i$, $i = 1, 2, \ldots, r$ are in the same block of $H$. Observe that $V(G_u)$, $u \in V(H)$ are precisely 1 imal nontrivial convex subsets of $G'$. ■

**Corollary:** A graph $G$ is (k,2)-convex only if there exists an n such that $(n-1)(n+2)/2$ $k$ $5 2n - 2$.

**Proof:** Follows from the fact that $|H_c|$ is minimum when $H$ is a path and maximum if it is a complet ■

Consequently, there is no uniconvex graph.

**Remark:** The converse of the above corollary is not true; since, if $k = 12$ or $13$, there is no (k,2) graph.

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