# GEODESIC ITERATION NUMBER

K.S. PARVATHY and A. VIJAYAKUMAR[1]

Department of Mathematics,
St. Mary's College,
Trichur - 680 020
India

[1]Department of Mathematics,
Cochin University of Science and Technology,
Kochi - 682 022
India

**Abstract :** In this paper we consider the geodesic iteration number and observe that in contrast with the minimal path iteration number, there are interval monotone graphs G and $S \subseteq V(G)$ with $|S| = 3$ such that gin (S) is arbitrarily large. However, join hull commutativity (JHC) property puts a bound on gin (S). We also prove that, if G is a geodetic, JHC graph then gin (G) = 1. These concepts play a significant role in convexity of graphs.

## 1. INTRODUCTION

We consider only finite, simple, connected graphs G. For the vertex set V(G) of G, several notions of path convexities have been studied. The most widely discussed are the geodesic convexity [19] and the minimal path convexity [9]. Different aspects of these types of convexities, such as evaluation of convex invariants [3, 11, 14], separation properties [2, 8], convex geometries [10], interval monotonicity [15, 16], convex simple graphs [1, 6, 12, 13, 17, 18] etc. have been well studied.

Let $S \subseteq V(G)$. If, for every $u, v \in S$, $I_d(u, v) = \{z \in V(G) : z$ is on some u-v shortest path$\} \subseteq S$, then S is geodesically convex or d-convex. If $I_d$ is replaced by $I_m(u, v) = \{z \in V(G) : z$ is on some u-v minimal path$\}$, then S is minimal path convex or m-convex. The $I_d$ and $I_m$ are called the geodesic interval and minimal path interval, respectively. Since, the convex sets are induced by suitable intervals, these graph convexity spaces are interval convexity spaces and hence have $\leq 2$ [5, 20]. For convenience, $I_d$ will be denoted

Mulder [16] has observed that the intervals need not be convex. A graph G is interval monotone if all its geodesic intervals are convex. Trees and Hypercubes $Q_n$ are examples of such graphs and $K_{2,3}$ is not interval monotone. The study of interval monotone graphs with m-convexity has been recently initiated in [15].

Let $\mathscr{C}$ denote the convex subsets of V(G). Then (G, $\mathscr{C}$) is join hull commutative (JHC) if for any $C \in \mathscr{C}$ and $v \in V(G)$, $Co(C \cup \{v\}) = \cup \{Co(\{c,v\}) : c \in C\}$, where Co(A) denotes the convex hull of A. JHC property in graphs is discussed in [7] and a detailed study of JHC property in abstract convexity spaces is in [20].

Also, (V(G), I) has Peano property if for any a,b,c,u,v of V(G) such that $u \in I(a, b)$, $v \in I(c, u)$, there is a $w \in I(b, c)$ such that $v \in I(a, w)$. It is known that the graphs associated with the five regular polyhedra have this property.

In this paper we study the concept of geodesic iteration number for graphs which are interval monotone and have JHC property.

An excellent survey of several aspects of convexity spaces and convexity in graphs is in [20].

## 2. MAIN RESULTS

**Definition 2.1[4]:** Let G = (V, E) be a connected graph and $S \subseteq V(G)$. Then, the closure of S, (S) = $\{x : x$ is on some shortest path connecting vertices of S$\}$. Define $S^k$ recursively as follows: $S^1 = (S)$, $S^k = (S^{k-1})$ for $k > 1$. The geodesic iteration number gin (S) is the smallest number n such that $S^n = S^{n+1}$. The geodesic iteration number gin (G) is defined as max $\{$gin (S) : $S \subseteq V(G)\}$. Replacing the shortest path by the minimal path, we get the notion of min (S) and min (G).

**Example :** Let k be any positive integer and G be the graph obtained by taking the sequential join of (k+1) copies of $K_2$, then gin(G) = min (G) = k.

In fact, gin(G) measures how many steps one has to perform to get G as the convex closure of a subset. In general not much can be said about the gin of a graph, so conditions on the convexity are required.

It is known that any graph with m-convexity has JHC property. Also, G is interval monotone with m-convexity if and only if min (G) = 1 [15].

For graphs with geodesic convexity, it is necessary that G is interval monotone and JHC in order that gin (G) = 1. But, it is not sufficient. For, let G be $Q_3$ labelled as in Fig.1.

Take S = $\{a_2, b_1, d_1\}$. Then $S^1 = V(G) - \{c_2\}$ and $S^2 = V(G)$. Hence gin (G) $\neq$ 1.

It is observed that if G is interval monotone but not JHC, there are graphs G and $S \subseteq V(G)$ with $|S| = 3$ such

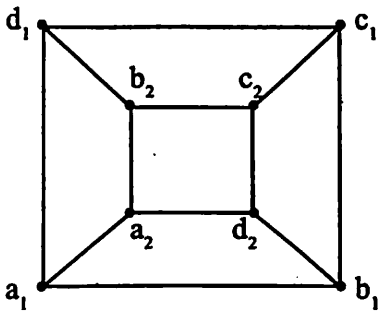


Fig.1

that gin (S) is arbitrarily large. But, JHC property plays a crucial role and we have,

**Theorem 2.1:** Let G be a JHC, interval monotone graph and let $S \subseteq V(G)$. Then gin $(S) \le k$, where k is such that $k-1 < \frac{\log |S|}{\log 2} \le k$.

**Proof :** Let $S = \{a_1, \ldots\ldots, a_n\}$,

$C_1 = Co(\{a_1, \ldots\ldots, a_{\lceil n/2 \rceil}\})$ and

$C_2 = Co(\{a_{\lceil n/2 \rceil + 1}, \ldots\ldots, a_n\})$.

Then $Co(S) = Co(C_1 \cup C_2)$

$= \cup \{Co(\{c_1, c_2\}) : c_1 \in C_1, c_2 \in C_2\}$

$= \cup \{ I(c_1, c_2) : c_1 \in C_1, c_2 \in C_2 \}$

since G is JHC and interval monotone.

Hence $Co(S) = \cup \{ I(c_1, c_2) : c_1, c_2 \in C_1 \cup C_2 \}$

$= (C_1 \cup C_2)^1$.

Now let $C_{11} = Co(\{a_1, a_2, \ldots\ldots, a_{\lceil n/4 \rceil}\})$

$C_{12} = Co(\{a_{\lceil n/4 \rceil + 1}, \ldots\ldots, a_{\lceil n/2 \rceil}\})$

$C_{21} = Co(\{a_{\lceil n/2 \rceil + 1}, \ldots\ldots, a_{\lceil 3n/4 \rceil}\})$

$C_{22} = Co(\{a_{\lceil 3n/4 \rceil + 1}, \ldots\ldots, a_n\})$

Then $C_1 = Co(C_{11} \cup C_{12})$ and

$C_2 = Co(C_{21} \cup C_{22})$. Then as above,

$C_1 = (C_{11} \cup C_{12})^1$ and

$C_2 = (C_{21} \cup C_{22})^1$

Hence $Co(S) = ((C_{11} \cup C_{12})^1 \cup (C_{21} \cup C_{22})^1)^1$

$= (C_{11} \cup C_{12} \cup C_{21} \cup C_{22})^2$.

$= \{Co(\{a_1, a_2, \ldots\ldots, a_{\lceil n/4 \rceil}\}) \cup Co(\{a_{\lceil n/4 \rceil + 1}, \ldots\ldots, a_{\lceil n/2 \rceil}\}) \cup \ldots\ldots\}^2$

$= \{Co(\{a_1, a_2, \ldots\ldots, a_{\lceil n/2^2 \rceil}\}) \cup Co(\{a_{\lceil n/2^2 \rceil + 1}, \ldots\ldots, a_{\lceil n/2 \rceil}\}) \cup \ldots\ldots\}^2$

Proceeding like this,

$Co(S) = \{Co\{a_1, a_2, \ldots\ldots, a_{\lceil n/2^k \rceil}\} \cup Co\{a_{\lceil n/2^k \rceil + 1}, \ldots\ldots, a_{\lceil n/2^{k-1} \rceil}\} \cup \ldots\ldots \cup \{a_{\lceil (2^k - 1) n / 2^k \rceil + 1}, a_n\}^k$.

Now when $\lceil n / 2^k \rceil = 1$

$2^{k-1} < n \le 2^k$ and $Co(a_1, a_2, \ldots\ldots, a_{\lceil n/2^k \rceil}) = Co(a_1) = \{a_1\}$

$Co(S) = (\{a_1\} \cup \{a_2\} \cup \ldots\ldots \cup \{a_n\})^k$

$= (\{a_1, \ldots\ldots, a_n\})^k = S^k$

Hence, gin $(S) \le k$, where $2^{k-1} < n < 2^k$.

That is $k-1 < \frac{\log n}{\log 2} \le k$. where $n = |S|$.

The following discussion illustrates that the bound for gin (S) is sharp.

Let k be any integer and $n = 2^k$. Let $Q_n$ be the n-cube, vertices labelled with (0,1) valued n-tuples.

Let $\delta_i = (x_1, \ldots\ldots, x_n)$ where $x_i = 1$ and $x_j = 0$ for $j \ne i$ and $\delta_0 = (0, 0, \ldots\ldots, 0)$.

Then, $d(\delta_i, \delta_j) = 2$, for $i \ne 0$.

Let $S = \{\delta_1, \delta_2, \ldots\ldots \delta_n\}$.

If $\delta_{i,j} = (x_1, x_2, \ldots\ldots, x_n)$ where $x_i = x_j = 1$ and $x_k = 0$ for $k \ne i,j$, then $\delta_{i,j}$ is adjacent to $\delta_i$ and $\delta_j$.

Hence, $S^1 = \{\delta_0\} \cup S \cup N_2(\delta_0)$.

Now if $\delta_{i,j}, \delta_{k,\ell} \in N_2(\delta_0)$ be such that $i, j \ne k, \ell$, then $d(\delta_{i,j}, \delta_{k,\ell}) = 4$ and if $A = \{i, j, k, \ell\}$ and $\delta_A = (x_1, \ldots\ldots, x_n)$ where $x_i = x_j = x_k = x_\ell = 1$ and $x_m = 0$ for $m \notin A$, then, $\delta_A \in I(\delta_{i,j} \delta_{k,\ell})$.

Hence $S^2 = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup N_3(\delta_0) \cup N_4(\delta_0)$

$= \{\delta_0\} \cup S \cup N_2(\delta_0) \cup N_3(\delta_0) \cup N_{2^2}(\delta_0)$

Similarly, $S^3 = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots\ldots \cup N_{2^3}(\delta_0)$, and

$S^k = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots\ldots \cup N_{2^k}(\delta_0) = V(Q_n)$.

Hence gin (S) = k.

**Note :** If n is such that $2^{k-1} < n < 2^k$, in the above example, $S^{k-1} = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots\ldots \cup N_{2^{k-1}}(\delta_0)$

$S^k = \{\delta_0\} \cup S \cup N_2(\delta_0) \cup \ldots \cup N_{2^{k-1}}(\delta_0) \cup \ldots \cup N_n(\delta_0)$.

Therefore if n is such that $2^{k-1} < n \le 2^k$, gin $(S) = $ gin $(\{\delta_i\}) = k$.

**Definition [20].** Let X be an interval convexity space. An interval I(a, b) of X is decomposable if for each $x \in I(a, b)$, $I(a, x) \cup I(x, b) = I(a, b)$.

We shall now describe the conditions for gin(G) to be 1. It turns out that if an interval monotone, JHC graph has the additional property that the geodesic intervals are decompasable, then gin G = 1. But, we first prove,

**Theorem 2.2:** A graph G is geodetic if and only if all its intervals are decomposable.

**Proof:** Let G be a geodetic graph, $a, b \in V(G)$ and $x \in I(a, b) = ab$ where ab is the unique shortest path joining a and b. Then $ab = ax \cup xb$ and hence $I(a,b) = I(a, x) \cup I(x, b)$.

Conversely, assume on the contrary that G is not a geodetic. Let $P_1$ and $P_2$ be two distinct shortest path joining a, b of V(G). Suppose $x \in P_1$ and $y \in P_2$ be such that ax, ay $\in$ E(G). Observe then that, y $\notin$ I(a, x), y $\notin$ I(x, b) because ax $\in$ E(G) and d(x, b) = d(y, b).

**Theorem 2.3:** If G is a geodetic, JHC graph then gin (G) = 1.

**Proof :** Since G is geodetic, it is interval monotone. Also, since G is JHC, the geodesic interval operator satisfies the Peano property by a theorem in [20]. We shall denote by ab the shortest path joining a and b. To prove that gin (G) = 1, it suffices to prove that gin (S) = 1 for any S $\subseteq$ V(G) with |S| = 3.

So, let a, b, c $\in$ V(G), u $\in$ ab, and v $\in$ cu. It is enough to prove that v is in one of the intervals I (a,b), I (b,c) or I (a,c). Because G is geodetic I (a,b) = ab.

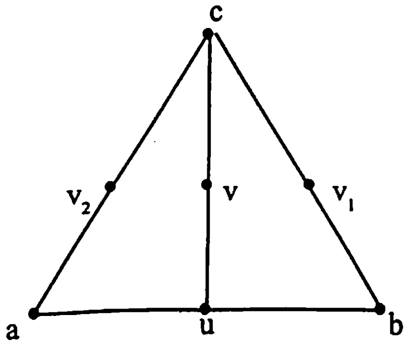


Fig.2

Assume without loss of genarality that d (c,v) = 1. Let d (a,c) = $\ell_1$ and d (b,c) = $\ell_2$. Now, by the Peano property, there are vertices $v_1 \in$ bc, $v_2 \in$ ac such that v $\in av_1 \cap bv_2$ Now, because d (a,c) = $\ell_1$, d (a,v) $\geq \ell_1$-1.

If d (a,v) = $\ell_1$-1 then d (a,c) = d (a,v) + 1 = d (a,v)+d (a,c) and hence v $\in$ ac.

So assume d (a,v) $\geq \ell_1$. If d (a,v) > $\ell_1$ then d $(a,v_1)$ = d (a,v) + d $(v,v_1) > \ell_1$ + d $(v,v_1)$. That is d$(a,v_1)$ > d (a,c) + d $(v,v_1)$ Now d $(a,v_1) \leq$ d (a,c) + d $(c,v_1)$ Therefore d $(v,v_1)$ < d $(c,v_1)$ and $\ell_2$ - d $(c,v_1) < \ell_2$ - d $(v,v_1)$; d $(b,v_1) < \ell_2$ - d $(v,v_1)$ d $(b,v_1)$ + d $(v,v_1) < \ell_2$ and so d (b,v) $\leq \ell_2$ - 1 and d (b,v) < $\ell_2$ - 1 is not possible and hence d (b,v) = $\ell_2$ - 1 and in this case v $\in$ bc.

Now assume that d (a,v) = $\ell_1$.

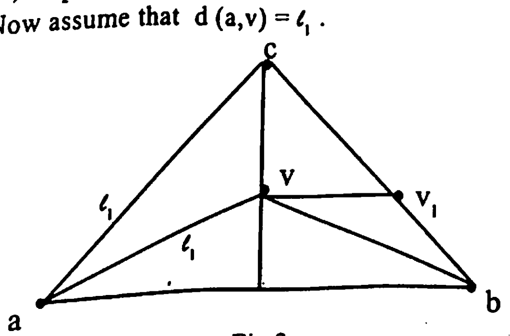


Fig.3

In this case d $(a,v_1) = \ell_1$ + d $(v,v_1)$. Now, if d $(c,v_1)$ > d $(v,v_1)$, then d $(b,v_1)$ + d $(v,v_1) \leq \ell_2$ - 1 and hence $v_1 \in$ bc.

So let d $(c,v_1) \leq$ d $(v,v_1)$. But d $(c,v_1)$ < d $(v,v_1)$ is not possible because $av_1$ is a shortest path containing v. Therefore d $(c,v_1)$ = d $(v,v_1)$. But this is again a contradiction because these give two distinct shortest paths connecting a and $v_1$. Hence the proof.

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