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ON THE CYCLIC CONVEXITY OF A GRAPH

K. S. PARVATHY* AND A. VIJAYAKUMAR**

**Department of Mathematics, St. Mary's College,* Trichur 680 020, *India*
***Department of Mathematics, Cochin University of Science and Technology,*
Cochin 682 022, *India*
e-mail: vijay@cusat.ac.in

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In this paper, a convexity for the edge set of a graph $G = (V, E)$ is defined. It is a matroid of rank $p - 1$ where $|V(G)| = p$ and its arity is not in general two. The classical convex invariants are also evaluated.

Key Words: Cyclic convexity; edgematroid; convexity parameters

1. INTRODUCTION

For the vertex set of a connected graph $G = (V, E)$ different types of convexities have been studied. All are path convexities and are induced by suitable interval operators, hence they are called interval convexity spaces. The arity of such convexity spaces is two [7]. Most widely studied are geodesic convexity and induced path (minimal path) convexity. Van de vel [7] has given a detailed account of combinatorial convexity theory in graphs and other discrete structures. Some other contributions on these lines are [1] to [6].

In this paper, we have two aims:

1. To extend the convexity notions to the edge set of a graph.
2. To study the properties of convexity spaces of arity greater than 2.

All basic graph theoretic and convex theoretic concepts are from [7].

In this paper we define cyclic convexity for the edge set of a graph. It is a matroid of rank $p - 1$. Classical convex invariants-Helly number, Radon number, Caratheodory number and exchange number are also evaluated.

2. CYCLIC CONVEXITY

finition 1 — [7] Let X be a non-empty set. A collection C of subsets of X is called a convexity $X \in C$, is closed for intersections and for chained unions. The pair (X, C) is called a convexity ice.

Members of C are called convex sets. For any set $S \subset X$ the convex hull of S is the smallest nvex set containing S and is denoted by $Co(S)$.

Definition 2 — [7] A convexity space X is of arity $<$ n if its convexity is determined by polytopes.

That is, a subset C of X is convex if and only if $Co(F) \subset C$ for each subset F of X of dinality at most n.

Let G be a graph with edge set E. Let C denote the collection of cycles and $\mathcal{C}_c$ denote the lection of chordless cycles of G and let $E = \{S \subset E : C \in \mathcal{C}$ and $|C \cap S| > |C| - 1$ implies $C \subset S\}$ and $\mathcal{E} = \quad \subset E : C \in \mathcal{C}_c$ and $|C \cap S| > |C| - 1$ implies $C \subset S$.

Lemma 1 — $E = \mathcal{E}_c$.

PROOF: It is obvious that $\mathcal{E} \subset \mathcal{E}_c$. Conversely, suppose that $S \in \mathcal{E}_c$ and let C be the cycle of gth k such that $|C \cap S| = |C|$, then it is clear that $C \subset S$. If $|C \cap S| = |C| - 1$, suppose that $\cap S = C - \{e\}$ and C is not induced. Otherwise $C \in \mathcal{C}_c$ and $C \subset S$ is clear. Assume that all cycles C' with $|C'| < |C|$ and with $|C' \cap S| > |C'| - 1$, $\quad \subset S$. Now, any chord e' of C ;ether with some path of $C - \{e\}$ forms a cycle C' of length $k' < k$ with $|C' \cap S| > \quad - 1$, ich implies that $e' \in S$. Thus, all chords of C are in S. The edge e belongs to some chordless cle C'' induced by some of the vertices of C with $|C \quad| < |C|$. Also $|C'' \cap S| > |C''| - 1 \Rightarrow$ $' \subset S \Rightarrow e \in S \Rightarrow C \subset S \Rightarrow S \in S$. Hence, $\mathcal{E}_c \subset E$.

Definition 3 — The collection $\mathcal{E}$ forms a convexity on E and is called the cyclic convexity.

Note: This convexity space is denoted by $(G, \mathcal{E})$.

In other words, $S \subset E$ is convex if and only if for any cycle C in G, $C \subset S$ whenever $C - e \subset S$ any $e \in C$. For any $S_0 \subset E$, $Co(S_0) = \{e' : \quad \in S_0$ or $C - e' \subset S_0$ for some cycle C in $G\}$.

Examples —

1. For a tree T every subset of $E(T)$ is trivially convex.

2. In the graph G.

Now, we shall consider a generalization of the notion of geodesic iteration number of an interval ivexity space to a convexity space of arity greater than two.

Definition 4 — Let X be a convexity space of arity n, $n > 2$ and $S \subset X$. The closure of S, toted by (S) is defined as $(S) = \bigcup\{Co(F) : F \subset X, |F| < n\}$. S^m is recursively defined as, $= (5), S^m = (S^m \quad)$. The smallest positive integer m such that $S^{m+\cdot} = S^m$ is called the ation number of S. The iteration number of X is defined to be the maximum iteration number of $S \subset X$.

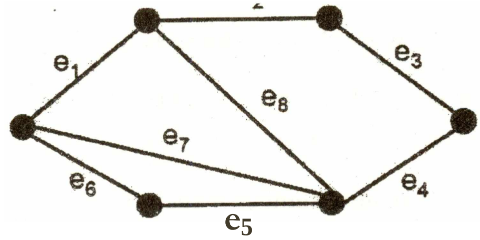

FIG. 1. $\{e_1, e_2, e_3\}, \{e_2, e_5, e_8\}$ **are convex, but** $\{e_5, e_6, e_8\}$ **is not convex.**

Lemma **2 —The iteration number of** $(G, \mathcal{E})$ **is 1.**

PROOF: Let $S \subset E$. It suffices to prove that $S^2 \subset S^1$. Let e $\in S^2$. Then, there is $C \in \mathcal{C}_c$ such that $C - \{e\} \subset S^1 = (S)$.

Let e' $\in C - \{e\}$. Then $E(C) - \{e, e'\}$ is convex. Hence e' $\notin C_0(E(C) - \{e, e'\})$. Now, $e' \in S^1$ implies that there exist a cycle $C' \in \mathcal{C}_c$ such that $C' - \{e'\} \subset S$. Then, $C - \{e'\} \cup C' - \{e'\}$ is a cycle C'' in C such that $C'' - \{e\} \subset S$, which implies that e $\in S^1$. Thus $S^1 = S^2$ and so the iteration number is 1.

Theorem 1 — The arity of $(G, \mathcal{E})$ *is one if G is a tree and is one less than the size of a largest chordless cycle in G, otherwise.*

PROOF: If G is a tree, then there is nothing to prove. Now, let k be the size of a largest chordless cycle C in G. It is clear that the arity is at most $k - 1$. On the other hand, let $S = E(C) - \{e\}$ for some edge e of C and suppose $S' \subset S, S' \neq S$. Then for any $C' \in \mathcal{C}_c$, $|C \cap S| < k - 2$. Therefore, $S' \in \mathcal{E}$, but $S \notin \mathcal{E}$. Thus, the arity is greater than $k - 2$ and so the arity of $(G, \mathcal{E}) = k - 1$.

Note: In literature, there are not many examples of convexity spaces of arity greater than two.

Definition **5 — [7] A convexity space** X **is a matroid if it satisfies the exchange law, for any convex set** $C \subset X$ **and a,** $b \in X - C$, $a \in Co(C \cup \{b\})$ **implies that** $b \in Co(C \cup \{a\})$.

Definition **6 —A non empty set** F **of** X **is convexly independent if** $x \notin Co(F - \{x\})$ **for each** $x \in F$.

Note: If X is a matroid, there exists a maximal independent subset of X called a basis and its cardinality, the rank.

Theorem **2** — *If G is a connected graph of order p, then* $(G, \mathcal{E})$ *is a matroid of rank* $p - 1$.

PROOF: It follows that $(G, \mathcal{E})$ is a matroid, by the definition of cyclic convexity. Now, let T be a spanning tree of G and let $F = E(T)$ and let e $\in F$.

Claim — $e \notin Co(F - \{e\})$.

$e \in Co(F - \{e\})$ implies that there is a cycle $C \in \mathcal{C}$ such that $C - \{e\} \subset F - \{e\}$ which implies $C \subset F$, a contradiction. Therefore, F is independent. Hence, rank of $(G, \mathcal{E}) \geq p - 1$. Now, if $|F| > p$ then F contains a cycle and hence it is convexly dependent. So, the rank is $p - 1$. ☐

ıe convex invariants of .a convexity space X are defined as follows [7].

The **Helly** number of X is the smallest number n such that $\bigcap\{Co(F-\{a\}) :$ a E F$\}$ (that F is **Helly** *(H—)* dependent) for all F C X with $|F| >$ n.

The Radon number of X is the smallest number n such that each F C X with $|F| >$ n can be ırtitioned into S1 and S2 such that $Co(S_1) \cap Co(S_2)$ ϕ. (that is, F is Radon *(R—)* dependent).

The **Caratheodory** number of X is the smallest number n such that $Co(F) = \bigcup\{Co(F-\{a\})$ E F$\}$ (that is, F is **Caratheodory** *(C--)* dependent) for all F C X with $|F| >$ n.

The exchange number of X is the smallest number n such that for F C X, $|F| >$ n and p E F, o$(F-\{p\})$ $\bigcup\{Co(F-\{a\}) :$ a E F$\}$ (that is, F is exchange *(E—)* dependent).

***Theorem* 3** — *Let G be a connected graph of order p, then the* **Helly** *number of (G, E),*

PROOF: Let T be a spanning tree of G and $F = E(T)$. We prove that F is **Helly** independent.

Suppose $\bigcap\{Co(F-\{e\}) :$ e E F$\}$ ϕ. Let e E F, E $Co(F-\{e\})$ and e' F. Then, ere is a cycle C in G such that $C -$ $\subset$ $S = F - \{e\}$.

Now, suppose that e' E $\bigcap\{Co(F-\{e\}) :$ e E F$\}$ where $e' = uv$. Then e' e for any $e \in F$. If e2 E F then there exist cycles C_1 and C2 in G such that $C_1 -$ $\subset$ $F -$ el and C2 $-$ $\subset$ $F -$ e2. ıen the cycle C comprising the paths $C_1 - e'$ and C2 $- e'$ is in T which is a contradiction.

Hence $\bigcap\{Co(F-\{e\}) :$ e E F$\} = \phi$. So F is **Helly** independent. Therefore $h(G) > p-1$.

Now, suppose $|F| > p$. Then F contains a cycle C in G. Then e' E $Co(F-\{$e'$\}$ for any e' E C ıd hence e' E $Co(F-\{e\})$ for any e E F, which implies that C C $\bigcap\{Co(F-\{e\}) :$ e E F$\}$. ıerefore, F is **Helly** dependent and so $h(G) < p$. Thus, $h(G) = p - \mathbf{1}$.

***Theorem* 4** — *The* **Caratheodory** *number of* $(G, \mathcal{E})$ *is given by,* $c(G) = 1$ *if G is a tree and* $irc(G) - 1$ *otherwise, where* $Circ(G)$, *the circumference of G is the length of a largest cycle in*

PROOF: If G is a tree, then every subset of E is convex. So for each F C E with $|F| > 1$, we ıve $Co(F) = F = \bigcup\{F-\{a\} :$ a E F$\}$ $\bigcup\{Co(F-\{a\}) :$ a E F$\}$. Hence $c(G) = 1$.

Now, let C be a longest cycle of length k and $S = E(C)$. Let $C =$ al $- e_1 -$ a2 $-$ e2 $- \ldots$ $_{-1} - a_k - e_k - a_1$, where a_i s are vertices and e_i s are edges. Let Si $= S - \{e_i\}, i = 1, 2, \ldots, k$.

Claim —*Si* is C-independent.

Here $Co(S_i) = S$ because $|C \cap S_i| = |S_i| = |C| - 1$. We have $Co(S_i) = E(C) = S$. But $Co(S_i - e)$ for any Si. Hence $S = Co(S_i)$ $\bigcup\{Co(S_i - e) : e$ E Therefore S_i is ·independent.· So $c(G) > |S_i| = |S|$ $1 = Circ(G) - 1 = k — 1$.

Now, if $F \subset E, |F| > k$ and if $e \in Co(F)$ then there is a cycle C in G such that $e \in C$ and $C - e \subset F$.

If $C - e \neq F$, there is an edge $e' \in F$ which is not in C. Then $C - e \subset F - e'$ and hence $e \in C = Co(C - e) \subset Co(F - e')$. Thus $Co(F) \subset \bigcup\{Co(F - e) : e \in F\}$.

If $C - e = F$ then $|C| = |F| + 1$, which is a contradiction because $Circ(G) = k$.

Theorem 5 — Let G be a connected graph of order p, then the Radon number of (G, E), $r(G) = p - 1$.

Theorem 6 — *Let G be a connected graph. The exchange number of* $(G, \mathcal{E})$, $e(G) = 2$, *if G is a tree or a cycle and is* $max\{Circ(G - v) : v \in G\}$, *otherwise.*

PROOF: *Case I* — Let G be a tree. Then every subset F of $E(G)$ is convex. If $|F| < 2$, let $F = \{e_1, e_2\}$. Then $F - \{e_1\}$ is not a subset of $F - \{e_2\}$. Hence F is E-independent. If $|F| > 3$, let $F = \{e_1, e_2, \cdots e_n, p\}$, $n > 2$. Then $Co(F - \{p\}) = F - \{p\} = \{e_1, e_2, \cdots e_n\} \subset \bigcup\{F - \{e_i\} : i = 1, 2 \ldots\}$. So, $Co(F - \{p\}) \subset \bigcup\{F - \{e\} : e \neq p, e \in F$.

Case II — Let G be a cycle. Then either $F = E$ or F has no subset comprising a cycle. If $F = E, Co(F - \{e\}) = F$ for each $e \in F$. If $F \neq E$, since F has no subset comprising a cycle, each proper subset of F is convex, so that the proof is same as that of Case I. Thus, in both these cases $e(G) = 2$.

Case III — Let G be a graph having a cycle C and a vertex $v \notin C$.

Assume that C is a largest cycle such that there is a vertex $v \notin C$. Let $C = a_1 - e_1 - a_2 - e_2 - \cdots - a_{n-1} - e_{n-1} - a_n - e_n - a_1$ where a_is are vertices and e_i s are edges between a_i and a_{i+1}. Let e be any edge incident on v and let $S = \{e_1, e_2, \ldots e_{n-1}, e\}$. Then $e_n \in Co(S - e)$, but $e_n \notin Co(S - e_i)$ for $i \neq n$. Hence S is E-independent and $e(G) > n$. Now, if $S = \{e_1, e_2, \ldots\}$, $m > n + 1$, let $e \in Co(S - e_i)$ for some i. Then there is a cycle C in G such that $e \in C$ and $C - e \subset S - e_i$. If $C - e = S - e_i, |C| = |S| > n + 1$, which is a contradiction. Hence there is some $e_j \in S - e_i$ such that $e_j \in C$ and so $e \in Co(S - e_j)$. Therefore, S is E-dependent and $e(G) < n + 1$. Thus $e(G) = n$.

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