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UNIFORMLY SEGREGATED TREES

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Abstract. In this paper family of Uniformly Segregated Trees (UST) and some of their properties are studied. The formula to find number of vertices for each k-segregated tree is found. The increase in the number of vertices in $S_kT_{:.}$ as of vertices of degree m increase is also determined. The method to construct higher (with regard to maximum degree) k-segregated trees from the lower k-segregated trees is discussed.
Subject Classification: 05C 12
Key words: Uniformly segregated tree, k-segregated tree, totally segregated tree

1. Introduction

A Tree is a connected acyclic graph. The number of edges incident to a vertex is degree of that vertex. If d1, $d_2, \ldots d_n$ are degrees of vertices of a graph a sequence of non negative integers $d_1, d_2, \ldots, d_{am}$ is a graphic sequence or a degree sequence. A graph is regular if all its vertices have the same degree. The eccentricity $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G i.e., $E(v) = \max_{v_i \in G} d(v, v_i)$. A vertex with minimum eccentricity in a graph G is called center of G. The eccentricity of a center (which is the distance from the center of the tree to the farthest vertex) in a tree is defined as the radius of the tree. Order of G $(0(G))$ is number of vertices in a graph G. In [1] a connected graph was defined to be highly Irregular if each of its vertices is adjacent only to vertices with distinct degrees. In [2] Jackson and Entringer extended this concept by considering those graphs on which every two adjacent vertices have distinct degrees and these graphs are named as totally segregated.

S_1T_3

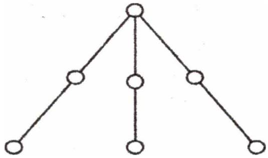


Figure 1.

S_3T_4

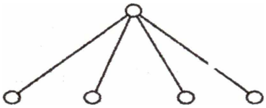


Figure 2.

2. Uniformly segregated trees (UST)

DEFINITION 2.1. **A** ***Totally Segregated Tree T is called Uniformly Segregated Tree (UST) if*** $|d(u) - d(v)|$ ***is a nonzero constant*** $\forall uv \in E(G)$. ***In particular if*** $|d(u) - d(v)| = k$, ***T is called k-segregated tree.***

PROPOSITION 2.2. ***Let T be a k-segregated tree with*** $\Delta(T) = n$, ***then*** $n \equiv 1 \bmod k$.

Proof **Let u be a vertex of degree n in** T **and v be any pendant vertex. On any u-v path, the degree of vertices increase or decrease by** ***k, as*** **it passes from one vertex to the next. Hence,** $1 = \deg(v) = n +$ **multiple of k. Hence,** $n \equiv 1 \bmod k$. **0**

Remark. Smallest k-segregated tree (w.r.t. no. of vertices) with maximum degree n is denoted by S_kT_n. **In this case, the number of vertices of maximum degree n is 1. If the number of vertex of maximum degree is 1, that vertex is called root vertex.**

Example. S_1T_3 **Smallest 1-Segregated Tree with maximum degree 3 (Fig. 1).**

Example. S_3T_4 **Smallest 3-Segregated Tree with maximum degree 4 (Fig. 2).**

THEOREM 2.3. ***Cartesian product of two k-segregated graphs is k-Segregated.***

Proof **Let** G_1, G_2 **be two k-segregated graph. In the Cartesian product** $G_1 \times G_2$, (u_i, v_j) **and** (u_1, v_m) **are adjacent iff** $u_i = u_1$ **and are adjacent or** $v_3 = v_m$ **and** u_i, u_1 **are adjacent. If** v_j, v_m **are adjacent in** G_2, $|d(v_j) - d(v_m)| = k$ **and if** u_1 **are adjacent in** G_1, $|d(u_i) \quad d(u_j)| = k$ **since,** G_1, G_2 **are k-segregated graphs.**

Hence if (u_i, v_j) **and** (u_1, v_m) **are adjacent in** $G_1 \times G_2$; $|d(u_i, v_j) \quad d(u_l, v_m)| = k$. ***0***

3. Total number of vertices in k-segregated tree

PROPOSITION 3.1. *Total number of vertices in $S_k T_{1,}$ is*

$$1 + n + n(n \quad k \quad 1) + n(n \quad k - 1)(n - 2k - 1)$$
$$+ \cdots + n(n \quad k - 1)(n - 2k \quad 1) \ldots 3k \times 2k$$
$$+ n(n \; k — 1\}(n - 2k —1) \ldots 3k \times 2k \times k$$

Proof Consider $S_k T_{1,}$. In this tree, maximum degree is n. Since number of vertices having each degree is minimum, number of vertices of degree n is 1. Since degree of that vertex is n, n vertices are adjacent to the vertex of degree n. Since degrees of adjacent vertices are segregated by k, these n vertices are of degree n k. Hence, number of vertices of degree n k = n. Consider one vertex of degree n k. Out of its n – k neighbours, n $k - 1$ vertices are of degree n 2k and one vertex is of degree n $2k = n(n - k - 1)$. One vertex of degree n 2k has n – 2k – 1 neighbours of degree n – 3k and one vertex of degree n – k. Hence, the number of vertices of degree n $3k = n(n - k - 1)(n - 2k - 1)$, and so on.

Degree and number of vertices of $S_k T_{1,}$ is shown in the table:-

Degree of Vertices	Number of Vertices
n k	
n – 2k	$n(n \;\; k - 1)$
n – 3k	$n(n \quad k \; 1)(n \quad 2k \; 1)$
2k	$n(n \quad k - 1)(n - 2k \; 1) \cdots . \times 3k \times 2k$
k	$n(n - k - 1)(n - 2k - 1) \cdots \times 3k \times 2k \times k$

Remark. If the number of vertices are not minimum, excess number of vertices is to be included in the notation. In $S_k T_{1,}$, if one vertex of degree m is increased with out changing maximum degree n vertex of degree m 2k is changed to vertex of degree m and some more vertices of *degree* less than m, get added to make it k-segregated. We denote it by $S_k T_{1,,m(1)}$. For example, if in Si T4 one vertex of degree 4 is increased, the resulting UST is denoted by Si Teo. In the same way, if one vertex of degree 3 is increased in Si Tel), the resulting UST is denoted by $S_1 T_{4(1),3(1)}$.

Here, note that it is not possible to increase number of vertices of degree 1 and 1 + k alone with out increasing vertices of higher degree in k-segregated tree. It is decided according to number of vertices of degree 1 + 2k.

PROPOSITION 3.2. *Increase in the no. of vertices in $S_k T_n$ as no. of vertices of degree* *m increase by 1 is*

$$1 + m\ 1 + (m-1)(m-k-1) - 1 + ((m\ 1)(m\ k-1)-1)(m-2k\ 1)$$
$$+ ((m-1)(m\ k\ 1)-1)(m-2k-1)(m-3k\ 1) +$$
$$((m-1)(m-k-1)-1)(m\ 2k\ 1)\cdots \times 2k$$
$$+ ((m-1)(m\ k-1)-1)(m\ 2k\ 1)\cdots \times 2k \times k$$

Proof. **In general, degree and number of vertices of $S_kT_{1,}$ as follows, where $n \equiv 1$ mod k.**

Degree of vertices	Number of vertices
n k	
n – 2k	$n(n-k-1)$
$n-3k$	$n(n-k-1)(n-2k-1)$
2k	$n(n\ k-1)(n-2k-1)\cdots \times 3k \times 2k$
k	$n(n\ k-1)(n-2k\ 1)\cdots \times 3k \times 2k \times k$

In $S_kT_{1,}$, if the number of vertices of degree m is increased by 1, where m 1, 1 + k, newly formed vertex of degree m, generate m – 1 vertices of degree m – k and remove I vertex of degree m – 2k, since vertex of degree m – 2k is changed in to vertex of degree m.
One vertex among m neighbours of degree m – k of newly formed vertex of degree m was existed there before.
Hence, increased number of vertices of degree m $k = m - 1$.
Each newly generated vertex of degree m – k has m $k - 1$ vertices of degree m – 2k.
But I vertex of degree m 2k is removed, since vertex of degree m – 2k is changed to vertex of degree m.
Increased number of vertices of degree m 2k = (m 1)(m $k-1)$ --1
Each newly generated vertex of degree m – 2k has m – 2k – 1 vertices of degree m – 3k.
Increased number of vertices of degree m – 3k = ((m – 1)(m – $k-1)-1)(m-2k$ 1) and so on.
Hence, total increase of vertices

$$= 1 + m - 1 + (in-1)(m\ k-1) - 1 + ((m-1)(m-k\ 1)-1)(m-2k-1)$$
$$((m-1)(m-k-1)-1)(m\ 2k-1)(m\ 3k-I)\quad .$$
$$+ ((m\ 1)(m-k\ 1)-1)(m-2k-1)\cdots \times 2k$$
$$((m-1)(m\ k-1)-1)(m-2k-1)\cdots \times 2k \times k.$$

□

Remark. Total number of vertices in S_1T_n *is*

$$1 + n \quad n(n-2) + n(n-2)(n-3) + $$
$$+ n(n \; 2)(n \; 3)\ldots 3 \times 2 + n(n \; 2)(n-3)\ldots 3 \times 2 \times 1.$$

4. Construction of S_kT_n from S_kT_{n-1}

DEFINITION 4.1. *In S_kT_n, n branches are attached (joined by an edge) to root vertex of degree n. If we remove all edges adjacent to root vertex in S_kT_n it becomes disconnected, and each component, called major branch of S_kT_n. S_kT_n has n major branches. If one major branch is removed from S_kT_n, the resulting tree, called major bunch of major branches of S_kT_n. In a major bunch of major branches of S_kT_n the vertex whose neighbours has the same degree as that of the vertex, called sub root vertex. Number of vertices in major branch of S_kT_n is* $O(S_kT_n) - 1$ *and number of vertices in major bunch of major branches of S_kT_n is*

$$\frac{(O(S_kT_n) - 1)(n-1)}{n} + 1.$$

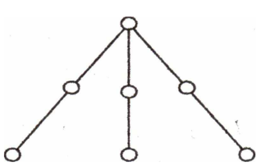
RESULT 4.2. *How to construct S_kT_n from S_kT_{n-1}.*

Procedure. Consider major bunch of major branches of S_kT_{n-1} and take its n copies. Add one vertex and make sub root vertex of each major bunch of major branches of S_kT_{n-1} adjacent to newly added vertex. Then we obtain S_kT_n.

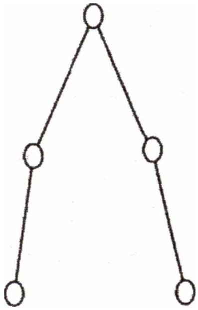
Remark. Major bunch of major branches of S_kT_n is major branch of S_kT_{n+1}.

EXAMPLE 4.3. *Construction of S_1T_4 from S_1T_3.*

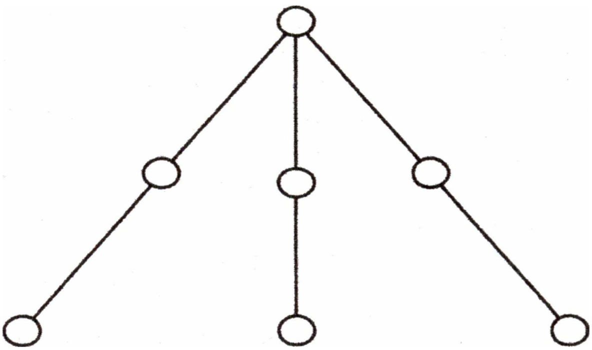
Consider S_1T_3



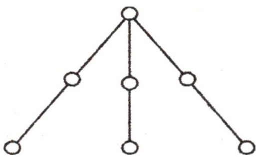
Consider major bunch of major branches of S_1T_3



Take four copies of it.



Add one vertex and make adjacent to the sub root vertex of 4 copies. Then we obtain S_1T_4.



PROPOSITION 4.4. *Another method to find increase in the number of vertices in* S_kT_n *as number of vertices of degree m increase by 1. In* S_kT_n, *if we increase number of vertices of degree m by 1 with out increasing maximum degree* n, *the vertex of degree n 2k is changed to m. The effect in number of vertices is that all vertices except sub root vertex in major bunch of major branches of* S_kT_n *is added, and all vertices except*

sub root vertex in major bunch of major branches of S_kT_{m-2k} is subtracted. Hence increase in number of vertices is

$$\frac{(O(S_kT_m)-1)(m-1)}{m}-\frac{(O(S_kT_{m-2k})-1)(m-2,k-1)}{m\ 2k}$$

When we simplify, we get the same formula as in Proposition 3.2.

PROPOSITION 4.5. *Number of vertices in 1-Segregated Tree is always odd.*

Proof Total number of vertices 1-Segregated Tree S_1T_n *is*

$$1+n+n(n\quad 2)+n(n-2)(n\ 3)+$$
$$n(n-2)(n\ 3)\ldots 3\times 2+n(n-2)(n-3)\ldots 3\times 2\times 1.$$

In this expression, all except the first three terms are even, since each term contains product of at least two consecutive integers. If n is odd, 2nd and 3rd terms are odd. Hence their sum is even. So, sum of terms except the first term is even. The first term is 1. Hence, the total number of vertices is even +1, which is odd. If n is even, all terms except first term is even. Hence, sum is even +1, which is odd. Increase in the number of vertices, as the number of vertices of degree m increase by 1, is

$$1+m\ 1+(m-1)(m-\quad -1+((m-1)(m-2)-1)(m-3)$$
$$+((m\ 1)(m-2)-1)(m-3)(m-4)+\cdots+\quad -1)(m-2)-1)(m-\quad \ldots$$
$$\times 2+((m-1)(m-2)-1)(m-3)\cdots\times 2\times 1$$

From this expression, we can easily see that, this is always even for any m.

In this expression, all except the first four terms are even, since each term contains product of at least two consecutive integers. If m is odd, the first four terms are odd. Hence, their sum is even. If m is even, 2nd and 4th terms are odd, and their sum is even. 1st and 3rd terms are even and their sum is even. Hence, increase in no. of vertices is always even. Hence, total number of vertices in 1-Segregeted tree is sum of odd and even which is odd. Hence the result. 0

PROPOSITION 4.6. *Total number of vertices in 1-segregated tree is of the form* $3+4k$, $k=0,1,2,\ldots$ *or* $1+4k$, $k=5,6,7,\ldots$.

Proof Let maximum degree $A=2$, only one 1-segregated tree exists with $A=2$, and its order is 3. Let $A=3$, minimum number of vertices of 1-segregated tree with $A=3(S_1T_3)$ is 7. Increase of number of vertices of degree 1 and 2 is not possible. When we increase number of vertices of degree 3, number of vertices increase in arithmetic progression with common difference 4. Hence, total number of vertices in that 1-segregated tree is $7+4k$, $k=0,1,2,\ldots$. Consider 1-segregated tree with $A=4$, minimum number of vertices of 1-segregated tree with $A=4$ is 21. When we increase number of vertices of degree 3, number of vertices increase in arithmetic progression with common difference 4. Hence, total number of vertices in that 1-segregated tree is 21 +4k, $k = 0, 1, 2, \ldots$ On higher level ($\Delta = 5, 6, \ldots$) total number of vertices in 1-segregated tree is an odd number greater than 21. As in Proposition 3.1, number of vertices in 1-segregated tree is always odd. $7 + 4k$, $k = 0, 1, 2, \ldots$ includes all odd number greater than or equal to 21. It is not possible that 1-segregated tree have 5, 9, 13, 17 as its order since $7 + 4k$, $k = 0, 1, 2, \ldots$ does not include 5, 9, 13, 17. Hence, total number of vertices in 1-segregated tree is $3 + 4k$, $k = 0, 1, 2, \ldots$, or $21 + 4k$, $k = 0, 1, 2, .$. That is total number of vertices is $3 + 4k$, $k = 0, 1, 2, \ldots$ or $1 + 4k$, $k = 5, 6, .$. Hence the result. 0

Remark. Here, we summarise important facts about uniformly segregated trees.
Note i: P_3 the is only path which is uniformly segregated.
Note ii: All stars are uniformly segregated trees.
Note iii: Radius of $S_kT_{i,}$ = half of diameter of $S_kT_{i,}$.

PROPOSITION 4.7. *Radius of $S_kT_{i,}$ is*

Prof. In $S_kT_{i,}$, vertex of degree n is center and $n \equiv 1 \mod k$. *i.e.*, $n = sk + 1$. In $S_kT_{i,}$, vertex of degree $n - k$ *is* at distance 1 from the center, vertex of degree n 2k is at distance 2 from the center, *etc.* Vertex of degree 1 ($= n \quad sk$) *is* at distance s from the center. Pendent vertex is farthest vertex from the center. Hence radius of $S_kT_{i,}$ is n-1 . 0 1

5. Application

Water supply network can be easily represented by a graph $G = (V, E)$ in which V corresponds to n nodes and E corresponds to pipes of water system. If we design water supply network in $S_kT_{i,}$ pattern, we can fix number of channels from the source point, which is vertex of maximum degree in the graph, so as to get the required number of destination points, at different distances from the source point.

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REFERENCES

[1] Y. Alavi, G. Chartrand, F.R.K. Chung, P. Erdos, R.L. Graham and O.R. Oilermann, Highly irregular graphs, J. Graph Theory (1987) 235-249

[2] D.E. Jackson and R. Entringer, Totally Segregated Graphs, Congress. Numer. 55 (1986) 159-165

Yousef Alavi, Alfred J. Boa's, Gary Chartrand, Ortrud R. Oellermann and Paul Erdos, K-path irregular graphs, Congressus Numerantium 65 (1988), 201-210

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Application, MacMillan, London (1979)

[5] F. Harary, Graph Theory, Addison-Wesley (1969)

[6] Douglas B. West, Introduction To Graph Theory, Second Edition, Prentice Hall of India, New Delhi (2006)