*Bulletin of Kerala Mathematics Association*
*Vol. 13, No.1, (2016 June)* 75-79

# COMMON FIXED POINT THEOREMS IN GENERALIZED FUZZY METRIC SPACES

Sukanya K.P.*, Magie Jose**

***GHSS Chettiyankinar, Valakulam P.O., Malappuram*
***Kerala, India.***

*****Department of Mathematics, St.Mary's College***
***Thrissur, Kerala, India.***
sukanyakpisharody@gmail.cam,
srmargaretmary.cmc@gmail.com

**(Received on 19.11.2015, Accepted on 23.04.2016)**

**Abstract.** C.T. Aage and J.N. Salunke proved fixed point theorems in fuzzy metric spaces for occasionally weakly compatible self maps. Guangpeng Sim and Kai Yang proved fixed point theorems in generalized Q-fuzzy metric spaces for weakly compatible self maps.This paper presents common fixed point theorems in generalized Q-fuzzy metric spaces for occasionally weakly compatible self maps.
Key Words: Q-fuzzy metric space, generalised Q-fuzzy metric spaces, weakly compatible self maps, occasionaly weakly compatible self maps

## 1. Preliminary Notes

**DEFINITION 1.1.** *Let X be a nonempty set and let G : $X \times X \times X$ R+ be a function satisfying the following:*

- $G(x, y, z) = 0$ *if* $x = y = z$
- *O*< $G(x, x, y)$ *for all x, y* $\in$ *X with x* 4 *y*
- $G(x, x,$ $< G(x, y, z)$ *for all x, y, z in X with z* y
- $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ *(symmetry in all three variables)*
- $G(x, y,$ $G(x, a, a) + G(a, y, z)$ *for all x, y, z, a* E *X (Rectangle inequality).*

*Then the function is called a generalized metric, or, more specifically a G-metric on X and the pair $(X, G)$ is a G-metric space.*

*Example.* Let $(X, d)$ be ametricspace.Then $G : X \times X \times X$ -4 $R^+$ defined by $G(x, y, z) = d(x, y) + d(y, z) + d(x, z)$. Then $(X, G)$ is a G-metric space.

**DEFINITION** 1.2. *A fuzzy set A in X is a function with domain X and values in* [0, 1]

**DEFINITION** 1.3. *A binary operation* $* : [0,1] \times [0, 1] \rightarrow [0,1]$ *is a continuous t-norm if* $*$ *satifies the following conditions:*

- **is commutative and associative*
- **is continuous*
- $a * 1 = a$ *for all* $a$ E [0, 1]
- $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0, 1]$.

## 2. Q-Fuzzy Metric Spaces

**DEFINITION** 2.1. *A 3-tuple* $(X, Q, *)$ *is called a Q-fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and Q is a fuzzy set in* $X^3 \times (0, \infty)$ *satisfying the following conditions for each* x, y, z, a E X *and* $t, s > 0$:

- $Q(x, x, y, t) > 0$ *and* $Q(x, x, y, t)$ C $Q(x, y, z, t)$ *for all* x, y, z e X *with* z y
- $Q(x, y, z, t) = 1$, *for all* $t > 0$ *if and only if* $x = y = z$
- $Q(x, y, z, t) = Q(p(x, y, z), t)$ *(symmetry), where p is a permutation function*
- $Q(x, a, a, t) * Q(a, y, z, s) \leq Q(x, y, z, t + s)$
- $Q(x, y, z, .) : (0, \infty)$ -4 $[0, 1]$ *is continuous.*
- *Q-fuzzy metric space can be considered as a generalization of fuzzy metric space.*

**EXAMPLE** 2.2. *Let X is a non empty set and G is the G-metric on X. The t-norm is* $a * b = ab$ *for all* $a, b$ E $[0,1]$. *For each* $t > 0 Q(x, y, z, t) = t/t + G(x, y, z)$ *Then* $(X, Q, *)$ *is a fuzzy Q-metric.*

**LEMMA** 2.3. *If* $(X, Q, *)$ *be a Q-fuzzy metric space, then* $Q(x, y, z, t)$ *is non-decreasing with respect to t for all* x, y, z E X

**DEFINITION** 2.4. *Let* $(X, Q, *)$ *be a Q-fuzzy metric space. A sequence* $\langle xn \rangle$ *in X converges to a point x E X if and only if* $Q(x_{n}, X_{m}, x, t)$ -4 1 *as* n $\infty$, m $\infty$.

## 3. Occasionally Weakly Compatible (OWC) Maps

**DEFINITION** 3.1. *For each* $t > 0$. *It is called a Cauchy sequence if for each* $0 < E < 1$ *and* $t > 0$, *there exists* no E N *such that* $Q(X_{n}, x_{m}, x_{l}) > 1 -$ E *for each* l, m, n $>$

3. *A Q-fuzzy metric space in which every Cauchy sequence is convergent is said to be omplete.*

DEFINITION 3.2. *Let f and g be self mappings on a Q-fuzzy metric space (X, Q, $*$). Then the mappings are said to be compatible* $\lim_{n \to \infty}(fgx_n, gfx_n, gfx_n, t) =$ *for all* $t > 0$ *whenever* $\langle x_n \rangle$ *is a sequence in X such that* $\lim_{n \to \infty} fx_n =$ $\lim_{n \to \infty} gx_n = z$ *for some z in X.*

DEFINITION 3.3. *Let X be a set. Let f and g be self maps on X. A point x in X is called a coincidence point off and g if and only if* $fx = gx$. *In this case* $w = fx = gx$ *s called point of coincidence off and g.*

DEFINITION 3.4. *A pair of self mappings (1, g) is said to be weakly compatible if :hey commute at the coincidence points,that is if* $fu = gu$ *for some* $u \in X$, *then* $fgu = gfu$.

DEFINITION 3.5. *Two self maps f and g of a set are occasionally weakly compatible owc) iff there is a point x in X which is a coincidence point off and g at which f and commute.*

LEMMA 3.6. *Let X be a set, f, gowc self maps of X. If f and g have a unique point of coincidence,* $w = fx = gx$, *then w is the unique common fixed point off and g.*

THEOREM 3.7. *Let (X ,Q,*) be a complete generalized Q-fuzzy metric space and let .4, B, S and T be self mappings of X. Let the pair {A, S} and {B, T} be occasionally weakly compatible (OWC). If there exist a* $k \in (0, 1)$ *such that for every* $x, y, z \in X$ *and* $t > 0$

$$Q(Ax, By, Bz, kt) > \min\{Q(Sx, Ty, Tz, t), Q(Sx, By, Tz, t), Q(By, Ty, Tz, t), Q(Ax, Ty, Tz, t), Q(Ax, Ty, Bz, t)\}. \quad (3.1)$$

*Then there exists a unique common fixed point of A, B, S and T*

*Proof.* The pair of self mappings A, $S$ and $B$, $T$ be occasionally weakly compatible (OWC). So there exist points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. First claim that $Ax = By$. If not by inequality (3.1).

$$\begin{aligned} Q(Ax, By, By, kt) &\geq \min\{Q(Sx, Ty, Ty, t), Q(Sx, By, Ty, t), Q(By, Ty, Ty, t), \\ &\quad Q(Ax, Ty, Ty, t), Q(Ax, Ty, By, t)\} \\ &= \min\{Q(Ax, By, By, t), Q(Ax, By, By, t), Q(By, By, By, t), \\ &\quad Q(Ax, By, By, t), Q(Ax, By, By, t)\} \\ &\quad Q(Ax, By, By, t) \end{aligned}$$

which is a contradiction. Therefore $Ax = By$, i.e., $Ax = Sx = By = Ty$.

Let there exist another point z such that $Az = Sz$. Then by inequality (3.1) we have $Az = Sz = By = Ty$. Therefore $Ax = Az$. i.e., $w = Ax = Sx$ is the unique point of coincidence of $A$ and $S$. Then by Lemma 3.6 w is the common fixed point $A$ and $S$. Similarly there is a unique point z E $X$ such that $z = Bz = Tz$.

Assume that w z. We have

$$Q(w, z, z, kt) = Q(Aw, Bz, Bz, t)$$
$$> \min\{Q(Sw, Tz, Tz, t), Q(Sw, Bz, Tz, t), Q(Bz, Tz, Tz, t),$$
$$Q(Aw, Tz, Tz, t), Q(Aw, Tz, Bz, t)\}$$
$$= \min\{Q(w, z, z, t), Q(w, z, z, t), Q(z, z, z, t), Q(w, z, z, t), Q(w, z, z, t$$
$$Q(w, z, z, t)$$

Hence $z = w$ and z is a common fixed point of $A$, $B$, $S$ and $T$. To prove uniqueness. let z' be another common fixed point of $A$, $B$, $S$ and $T$. If z $x'$. We have

$$Q(z', z, z, kt) = Q(Az, B2, Bz, t)$$
$$\min\{Q(Sz', Tz, Tz, t), Q(Sz', Bz, Tz, t), Q(Bz, Tz, Tz, t),$$
$$Q(Az', Tz, Tz, t), Q(Az', Tz, Bz, t)\}$$
$$= \min\{Q(z', z, z, t), Q(z', z, z, t), Q(z, z, z, t),$$
$$Q(z', z, z, t), Q(z', z, z, t)\}$$
$$Q(z', z, z, t)$$

Which is a contradiction.Hence $z = z'$, ie z is a unique common fixed point of $A$, $B$. $S$ and $T$. 0

THEOREM 3.8. *Let (X, Q,*) be a complete generalized Q-fuzzy metric space and let A, B, S and T be self mappings of X.Let the pair* {A, S} *and* {B, T} *be occasionally weakly compatible (owc). If there exist a k* E (0, 1) *such that for every x, y, z* E *X and* $t > 0$

$$Q(Ax, By, Bz, kt) \geq \min\{Q(Sx, Ty, Tz, t), Q(Sx, By, Tz, t), Q(By, Ty, Tz, t).$$
$$Q(Ax, Ty, Tz, t), Q(Ax, Ty, Bz, t)\}). \quad (3.2)$$

*For all x, y, z* E *X and such that (t)* $> t$ *for* $0 < t < 1$ *and (1)* $= 1$ *Then there exists a unique common fixed point of A, B, S and T*

*Proof* The pair of self mappings {A, S} and {B, T} be occasionally weakly compatible (owc). So there are pointsx,$y$ E $X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not by inequality (3.2) we have

$$Q(Ax, By, By, kt) > \varphi(\min\{Q(Sx, Ty, Ty, t), Q(Sx, By, Ty, t), Q(By, Ty, Ty,$$
$$Q(Ax, Ty, Ty, t), Q(Ax, Ty, By, t)\})$$
$$= \varphi(\min\{Q(Ax, By, By, t), Q(Ax, By, By, t), Q(By, By, By, t),$$
$$Q(Ax, By, By, t), Q(Ax, By, By, t)\})$$

$$\varphi(\min\{Q(Ax, By, By, t), Q(Ax, By, By, t), 1, Q(Ax, By, By, t), Q(Ax, By, By,$$
$$> Q(Ax, By, By, t)$$

which is a contradiction.Therefore $Ax = By$, *i.e.*, $Ax = Sx = By =$ Ty.

Let there exist another point z such that $Az = Sz$.Then by inequality (3.2) we have Az $= Sz =$ By $=$ Ty. Therefore $Ax = Az$. *i.e.*, $w = Ax = Sx$ is the unique point of coincidence of A and $S$. Then by Lemma 3.6 w is the common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that z $= Bz = Tz$. Assume that w z. We have

$$Q(w,z, z, kt) = Q(Aw, Bz, Bz, t)$$
$$(\min\{Q(Sw, Tz, Tz, t), Q(Sw, Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Aw, Tz, Tz, t), Q(Aw, Tz, Bz, t)\})$$
$$\varphi(\min\{Q(w, z, z, t), Q(w, z, z, t), Q(z, z, z, t), Q(w, z, z, t), Q(w, z, z, t)\})$$
$$\varphi(\min\{Q(w, z, z, t), Q(w, z, z, t), 1, Q(w, z, z, t), Q(w, z, z$$
$$> Q(w, z, z, t)$$

Hence z $=$ w and z is a common fixed point of $A$, $B$, $S$ and $T$. To prove uniqueness, let $z'$ be another common fixed point of $A$, $B$, $S$ and $T$. If z z' We have

$$Q(z', z, z, kt) = Q(Az', Bz, Bz, t)$$
$$\varphi(\min\{Q(Sz, Tz, Tz, t), Q(Sz', Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Az', Tz, Tz, t), Q(Az', Tz, Bz, t)\})$$
$$= \varphi(\min\{Q(z, \quad z, t), Q(z' z, z, t), Q(z, \quad z, t), Q(z', z, z, t), Q(z' z, ;OD$$
$$\varphi(\min\{Q(z, z, z, t), Q(z' \quad z, t), 1, Q(z', \quad z, t), Q(z' z, z, t)\})$$
$$> Q(z', z, z, t)$$

Which is a contradiction. Hence $z = z'$, *i.e.*, $z$ is a unique common fixed point of $A$, $B$, $S$ and $T$. 0

## REFERENCES


[1] A. George, P. Veeramani: *"On some results in Fuzzy metric spaces"* (1994)

[2] Z. Mustafa, W. Shatanawi, M. Bataineh: *"Existence of fixed point results in G-metric spaces"* (2009)

[3] B.C. Dhage: *"Generalized metric spaces and topological structure"* (2000)

[4] C.T. Aage, J.N. Salunke: *"On fixed point theorems in fuzzy metric spaces"* (2010)

[5] G. Sun, K. Yang: *"Generalized fuzzy metric spaces with properties"* (2010)