C 61916 (Pages : 3) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2019

(CUCSS—PG)

Mathematics

MT 4E 14—DIFFERENTIAL GEOMETRY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

*Answer **all** questions.*
Each question carries weightage 1.

1. Describe the level set at $c = 1$ for the function $f(x_1, x_2) = x_1^2 + x_2^2$.

2. Sketch the vector field on $\mathbb{R}^2$ given by $X(p) = -p$ for $p \in \mathbb{R}^2$.

3. Show that the n—sphere $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$ is an n—surface.

4. Give an example of a connected 2—surface.

5. Define sphereical image of an n—surface.

6. Define vector field along a parametrized curve.

7. Define covariant derivative of a smooth vector field along a parametrized curve.

8. Let X be a vector field parallel along a curve α Show that $\|X\|$ is constant.

9. Show that $\nabla_{cv} f = c\nabla_v f$ for all smooth functions f and real number c.

10. Find $L_p(v)$ for $p = (0, 0, 1)$ and $v = (0, 1, 2)$ on the cylinder $x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$.

11. Find the length of the parametrized curve $\alpha : I \to \mathbb{R}^2$ given by $\alpha(t) = (t^2, t^3)$ for $t \in [0, 2]$.

12. Let $k_1(p) = 1$ and $k_2(p) = 1/2$ be principal curvatures of an n—surface S at p. Find the Gaussian curvature of S at p.

Turn over

13. Define parametrized n—surface.

14. Let $\phi(\theta, \psi) = (\cos\theta \sin\psi, \sin\theta, \sin\psi, \cos\psi))$ for $0 < \theta < 2\pi$ and $0 < \psi < \pi$. Describe ϕ^{-1}.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries weightage 2.

15. Sketch the gradient field of the function $f(x_1, x_2) = x_1 + x_2$.

16. Define n—surface and give an example of a 1—surface.

17. Find the spherical image of an n—plane.

18. With the usual notations prove that $(X \dot{+} Y) = (\dot{X} + \dot{Y})$.

19. Prove that geodesics have constant speed.

20. Let X' denote the covariant derivative of a vector field X along a parametrized curve α. Show that $(X \cdot Y)' = X' \cdot Y + X \cdot Y'$.

21. Find the curvature k of the plane curve $x_1 + x_2 = 1$.

22. Let C be a plane curve and $p \in C$. Show that $L_p(v) = cv$ for some real number c.

23. Describe the normal section of an n—surface.

24. Let U be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a smooth function. Show that $\phi : U \to \mathbb{R}^{n+1}$ defined by $\phi(p) = (p, f(p))$ is a parametrized n—surface in $\mathbb{R}^{n+1}$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage 4.

25. (a) Define integral curve of a vector field.

(b) Let X be a smooth vector field on an open set U in $\mathbb{R}^{n+1}$. Show that there is an integral curve α on X.

(c) Show that for the vector field given by $X(x_1, x_2) = (-x_2, x_1)$, the parametrized curve $\alpha(t) = (\cos t, \sin t)$ is an integral curve for X.

26. (a) Let f, g be smooth functions from an open set U to $\mathbb{R}$. Let $S = f^{-1}(c) \nabla f(q) \neq 0$ for all $q \in S$. Let $p \in S$ be an extreme point of g. Show that there exists a real number λ such that $\nabla_g (p) = \lambda \nabla f(p)$.

(b) Using Langrange multiplier find an extreme point on the unit circle $x_1^2 + x_2^2 = 1$ for the function $g(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$.

27. Prove that :

(a) $\alpha : I \rightarrow S$ is a geodesic on S if and only if α satisfies the differential equation

$$\ddot{\alpha}(t) + (\dot{\alpha}(t) \cdot \dot{N}(\alpha(t))) N(\alpha(t)) = 0$$

for all $t \in I$ where $N = \nabla f / \| \nabla f \|$.

(b) if α is a solution of the above differential equation then α is a curve on S.

28. Let V be a finite dimensional vector space with dot product and $L : V \rightarrow V$ be a self adjoint linear transformation. Let $S = \{v \in V : v : v = 1\}$ and $f : S \rightarrow \mathbb{R}$ be defined by $f(v) = L(v) \cdot v$. Show that

(a) If $v_0 \in S$ is a stationary point of f then v_0 is an eigen vector of L.

(b) If $v_0 \in S$ is an eigen vector of L then f is stationary at v_0.

(2 × 4 = 8 weightage)