C 60051 **(Pages : 4)** **Name..................................**

Reg. No..................................

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2019

(CUCBCSS)

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. State the division algorithm.
2. State the Fundamental Theorem of Arithmetic.
3. Give an example to show that $a^2 \equiv b^2 \pmod{n}$ does not imply $a \equiv b \pmod{n}$.
4. Define multiplicative functions.
5. If x is a real number, what are the possible values of $[x]+[-x]$?
6. Define Euler phi function.
7. Find the highest power of 5 dividing 1000 !.
8. Define subspace of a vector space V.
9. Find Span S where $S = \{(1, 0, 0)\} \subseteq \mathbb{R}^3$.
10. Define a linear transformation.
11. Give two different bases for $\mathbb{R}^2$.
12. Define null space of a linear transformation.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 *marks.*

13. Prove that the square of any odd integer is of the form $8k+1$ where k is an integer.
14. Prove that if $\gcd(a, b) = d$, then $\gcd\left(a/d, b/d\right) = 1$ where a, b are integers.

Turn over

15. Let $\gcd(a, b) = 1$. Prove that $\gcd(a + b, a - b) = 1$ or 2.

16. Determine all solutions of the Diophantine equation $56x + 72y = 40$.

17. Prove that if $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$ and $ac \equiv bc \pmod{n}$.

18. Solve the congruence $18x \equiv 30 \pmod{42}$.

19. Prove that if p is a prime, then $a^p \equiv a \pmod{p}$ for any integer n.

20. Prove that τ is a multiplicative function.

21. Prove that the intersection of two subspaces of a vector space V is again a subspace of V.

22. Check whether the vectors $(1, 1, 0)$ and $(2, 5, 3)$ and $(0, 1, 1)$ in $\mathbb{R}^3$ are linearly independent.

23. Let W be a subspace of a vector space V. Prove that dim W = dim V if and only if V = W.

24. Prove that $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(a, b) = (a + b, a - b, b)$ is a linear transformation.

25. Let V and W be vector spaces. Prove that if the linear mapping $f : V \to W$ is injective and $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent subset of V, then $\{f(v_1), f(v_2), \ldots\ldots, f(v_n)\}$ is a linearly independent subset of W.

26. Let V and W be vector spaces. Prove that the linear mapping $f : V \to W$ is injective if and only if $\text{Ker } f = \{0\}$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

27. Prove that $\sqrt{2}$ is irrational.

28. Prove that the sequence of primes is infinite.

29. Prove that the linear congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d \setminus b$ where $d = \gcd(a, n)$. If $d \setminus b$ prove that the congruence has d mutually incongruent solutions modulo n.

30. Use Chinese Remainder Theorem to find the smallest non-negative solution of the given system of congruences :
$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}.$$

31. If n and r are positive integers with $1 \leq r < n$, then the binomial co-efficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is an integer.

32. Let V be a vector space over a field F. Prove the following :

(a) $\lambda 0_V = 0_V \ \forall \lambda \in F$.

(b) $0_F x = 0_V \ \forall_x \in V$.

(c) If $\lambda x = 0_V$, then either $\lambda = 0_F$ or $x = 0_V$.

33. Let S and T be two non-empty finite subsets of a vector space V such that $S \subseteq T$. Prove the following :

(a) If T is linearly independent, then so is S. (b) If S is linearly dependent, then so is T.

34. Let V be a finite dimensional vector space. If G is a finite spanning set of V and if I is a linearly independent subset of V such that $I \subseteq G$, prove that there is a basis B of V such that $I \subseteq B \subseteq G$.

35. Let V and W be vector spaces of finite dimension over a field F. If $f : V \to W$ be linear, prove that $\dim V = \dim \operatorname{Im} f + \dim \operatorname{Ker} f$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 marks.

36. Let $N = a_m 10^m + a_{m-1} 10^{m-1} + ... + a_1 10 + a_0$ be the decimal expansion of the positive integer $N ; 0 \leq a_k < 10$ and let $S = a_0 + a_1 + ... + a_m$,

$T = a_0 - a_1 + a_2 - \ldots + (-1)^m a_m$. Prove the following :

(a) $9 \mid N$ if and only if $9 \mid S$.

(b) $11 \mid N$ if and only if $11 \mid T$.

(c) Use the results in (a) and (b) to show that 1571724 is divisible by both 9 and 11.

37. (a) If n is a positive integer and $\gcd(a, n) = 1$, prove that $a^{\phi(n)} \equiv 1 \pmod{n}$. Deduce that if p is a prime and p / a, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) For $n > 1$, prove that the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n \phi(n)$.

38. Prove the following :

(a) Let V be a vector space of dimension $n \geq 1$ over a field F. Then V is isomorphic to the vector space F^n.

(b) If V and W are vector spaces of the same dimension n over a field F, then V and W are isomorphic.

(c) A linear mapping is completely and uniquely determined by its action on a basis.

(2 × 13 = 26 marks)